Ph.D. Program in Physics  
The City University of New York

The hereon is a syllabus of and suggested references for the First Examination in Physics on the topic of Quantum Mechanics.

1. The Theory and Its Interpretation
   a) Early microscopic experiments and the failure of classical physics.
   b) The Schrodinger and Heisenberg Formulation of quantum mechanics.
   c) The basic postulates of quantum mechanics.

2. General Formalism and Description of Physical Phenomena
   a) Dynamical states, physical quantities and their measurement.
   b) Heisenberg uncertainty relations.
   c) Introduction to mathematical formulation:
      Hermitian Operators, dynamical variables, commutator algebra, expectation values, eigenfunctions, function space, time evolution of an operator, constants of motion, etc
   d) The Schrodinger equation: Boundary values and initial conditions, discrete and continuous solutions.

3. One Dimensional Problems
   a) Piecewise constant potentials.
   b) The Linear Harmonic Oscillator, annihilation and creation operators.

4. Classical Approximation and the WKB Method
   a) Classical limit of the Schrodinger equation, Ehrenfest’s theorem, applications.
   b) The WKB approximation, barrier penetration.

5. Angular Momentum and Spherically Symmetric Potentials.
   a) Angular momentum operators, eigenfunctions and eigenvalues.
   b) Spherical harmonics.
   c) Hydrogen atom and other examples.
   d) Two-body problem and physical examples.

   a) Hermitian and Unitary operators, commutation rules and algebra, unitary transformations.
   b) Matrix formulation, matrix algebra, dynamical variables as matrices.
   c) The eigenvalue problem as a basis change.
   d) Linear vector space, vectors and operators, projection operators.
e) Time evolution operators, propagators.
f) Schrödinger, Heisenberg, and interaction pictures.
g) Canonical quantization.

7. Spin and Identical Particles

a) Electron spin and Pauli Matrices.
b) Symmetrization postulates and their applications.
c) Spin-Orbit forces.

8. Atoms

a) Central field approximation.
b) Pauli principle and periodic table
c) L-S and j-j coupling.
d) Symmetric and antisymmetric wavefunctions...


a) Non-degenerate Perturbation theory.
b) The degenerate case.
c) Examples, the Stark and Zeeman effects.
d) The sudden approximation as a change in basis.
e) Introduction to variational approximation methods.


a) Rotation operators and angular momentum.
b) Symmetry properties and commutation laws.
c) The rotation group and its interpretation.
d) Addition of angular momenta.
e) Parity, Time reversal, Charge conjugation.

11. Time-Dependent Perturbation Theory

a) General Formalism and examples.
b) Discussion of Electromagnetic interaction.
c) Interaction of atoms with electromagnetic radiation.
d) Fermi’s golden rule.
e) Cross-sections and transition probabilities.
f) Exponential decay (Wigner-Weisskopf method).
g) Coulomb excitations, inelastic conditions.

12. Scattering Theory

a) One-dimensional scattering.
b) Partial wave expansion.
c) Green’s function and S-matrix.
d) Born approximation.
13. Semi-classical Radiation Theory
   a) The atom in an electromagnetic field.
   b) Absorption, induced emission, spontaneous emission.
   c) Dipole selection rules, forbidden transitions, intensities.

14. Relativistic Quantum Mechanics
   a) The Klein-Gordon-Fock equation.
   b) Free-particle Dirac Equation, non-relativistic limit.
   c) Electromagnetic interaction of Dirac particles.
   d) Hole theory and positrons.

References:

E. Merzbacher, *Quantum Mechanics*, 2nd ed., Wiley
G. Baym, *Lectures on Quantum Mechanics*, Benjamin

Additional topics suggested for U725 and U726 courses (not required for the first examination):

   a) The path integral formulation of Quantum Theory.
   b) Density matrices and quantum statistics
   c) Second quantization of the electromagnetic field.
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First Examination — QUANTUM MECHANICS
August 4, 1998

Do any 6 out of 7 problems. You must indicate which problems you want counted — if you do not do so then problems 1–6 will be graded. Each question is worth the same number of points, 17, for a total of 102 points. The numbers in brackets [ ] indicate the number of points each part is worth. Some Formulas are given at the end of the exam.
1. (a) Starting with the one dimensional Schrödinger time dependent equation in position space

\[ i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x)\psi(x, t) \]

derive that the time dependent equation for the momentum wave function

\[ \phi(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x, t)e^{-ipx/\hbar}dx \]

is

\[ i\hbar \frac{\partial \phi(p, t)}{\partial t} = \frac{p^2}{2m}\phi(p, t) + \int_{-\infty}^{\infty} K(p')\phi(p - p')dp' \]

where

\[ K(p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} V(x)e^{-ipx/\hbar}dx \]

(b) Specialize to the case where

\[ V(x) = -Fx \]

where \( F \) is a constant. Show that the above reduces to

\[ i\hbar \frac{\partial \phi(p, t)}{\partial t} = \frac{p^2}{2m}\phi(p, t) - i\hbar F \frac{\partial \phi(p, t)}{\partial p} \]

(c) Solve this equation by attempting a solution in the form

\[ \phi(p, t) = f(p - Ft)g(p) \]

to obtain an equation for \( g(p) \). Solve for \( g(p) \).

(d) Suppose that at time \( t = 0 \) the momentum wave function is given by \( \phi_0(p) \). Express \( f(p) \) in terms of the initial momentum wave function and hence write the general solution for \( \phi(p, t) \) in terms of \( \phi_0(p) \).
2. Consider the eigenfunctions and eigenvalues for a charged particle of charge \( q \) in a uniform (and constant) magnetic field in the \( z \) direction. The Hamiltonian is

\[
H = \frac{1}{2m} \left( p - \frac{q}{c} A \right)^2
\]

where \( A \) is the vector potential and related to the magnetic field \( B \) by

\[
B = \nabla \times A
\]

For this problem we consider a uniform constant magnetic field in the \( z \) direction,

\[
B = (0, 0, B)
\]

For this case

\[
A = (-yB, 0, 0)
\]

(a) [5] Write the three dimensional Schrödinger time independent equation for this problem in the Cartesian coordinate system.

(b) [6] Write the wave function as

\[
\psi(x, y, z) = e^{ikx + ikz} g(y)
\]

and obtain an equation for \( g(y) \).

(c) [6] Obtain the energy eigenfunctions and eigenvalues, expressing your results in terms of the eigenfunctions and eigenvalues of the ordinary one dimensional harmonic oscillator

\[
\left( \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 \right) u_n(\alpha x) = E_n u_n(\alpha x)
\]

where

\[
E_n = (n + \frac{1}{2})\hbar \omega \quad \alpha = \sqrt{\frac{m\omega}{\hbar}}
\]

and

\[
u_n(x) = N_n H_n(x)e^{-\frac{x^2}{2}}
\]

where \( H_n \) are the Hermite polynomials and \( N_n \) is the normalization constant.
3. All parts of this problem are one dimensional.

(a)[5] Prove the variational Principle. That is prove that
the expected value of the Hamiltonian operator with any wave function
is equal to or greater than the ground state energy for that Hamiltonian.

(b)[4] Use any method you want to prove the following: If a wave function
differs from the true wave function by first order then the energy will differ
by second order

(c)[5] Use the trial wave function

\[ u \sim \frac{1}{\alpha^2 + x^2} \]

to estimate the ground state energy of the Harmonic oscillator. The variational
parameter is \( \alpha \).

(d)[3] Indicate how and under what circumstances the variational principle
can be used to estimate the first excited state energy.
4. (a)[9] Starting with Schrödinger's time dependent wave equation, derive first order time dependent perturbation theory. In particular show that if a system is in the $i$th energy eigenstate of the unperturbed Hamiltonian, then the probability that a transition has taken place to state $j$ is given by

$$|c_j(t)|^2 = \frac{1}{\hbar^2} \left| \int_{-\infty}^{t} V_{ji}(t')e^{-i\omega_{ij}t'} dt' \right|^2$$

where

$$\omega_{ij} = \frac{E_i - E_j}{\hbar}$$

and where the $E_i$'s are the energy eigenvalues of the unperturbed Hamiltonian and $V_{ji}(t)$ are the matrix elements of the perturbation taken with the eigenfunctions of the unperturbed Hamiltonian.

(b)[8] A harmonic oscillator is in the ground state. Starting at $t = -\infty$ it is perturbed by the following Hamiltonian:

$$V(x,t) = \alpha \delta(x - vt)$$

where $\alpha$ and $v$ are constants. Calculate the probability that the oscillator will be in "the first excited state" at $t = \infty$. The ground and first excited states of the Harmonic oscillator are

$$u_0(x) = \left( \frac{\alpha^2}{\pi} \right)^{1/4} e^{-\alpha^2 x^2/2}$$

$$u_1(x) = \sqrt{2\pi} \left( \frac{\alpha^3}{\pi} \right)^{3/4} xe^{-\alpha^2 x^2/2}$$

where

$$\alpha = \sqrt{\frac{m\omega}{\hbar}}$$
5. In a 3-dimensional motion, the classical expression for angular momentum is \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \); thus the corresponding quantum mechanical angular momentum must be \( \hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} \). From its three components we can form the operator for the magnitude (squared) of angular momentum:

\[
L^2 = L_x^2 + L_y^2 + L_z^2
\]

(a)[6] Prove that \([L_x, L^2] = [L_y, L^2] = [L_z, L^2] = 0\). Give a physical interpretation of this result.

(b)[6] Prove that the raising and lowering operators \( L_{\pm} = L_x \pm iL_y \) do not affect the magnitude of the eigenstate of the total angular momentum, i.e. that if

\[
L^2 \left| E, L^2, L_z \right> = \lambda \left| E, L^2, L_z \right>
\]

then

\[
L^2 L_{\pm} \left| E, L^2, L_z \right> = \lambda L_{\pm} \left| E, L^2, L_z \right>
\]

(c)[5] The orbital angular momentum state of an electron in an atom is measured to be \( Y_{\pm 1}(\theta, \phi) \). In the spin one representation of the angular momentum operator this means that

\[
L^2 \left| 1, -1 \right> = 2\hbar^2 \left| 1, -1 \right>, \quad L_x \left| 1, -1 \right> = \hbar \left| 1, -1 \right>
\]

What is the probability that it is measured later to in a state where the x-component of its angular momentum is equal to \( \hbar \)?

(Hint: Diagonalize \( L_x^{(1)} \) to find the state where its eigenvalue is equal to \( \hbar \).)

Note: The spin-1 representation of the angular momentum is

\[
L_x^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z^{(1)} = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
6. The Hamiltonian describing two identical spin-$\frac{1}{2}$ particles is given by

$$H = A(\sigma_z^{(1)} + \sigma_z^{(2)}) + B\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}$$

where $A$ and $B$ are some constants, and

$$\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. The states of the Hamiltonian are written in the Dirac notation as $|\uparrow\rangle$, as an example.

(a)[4] Find the eigenstates of $S_z$ and show that they are also eigenstates of total spin angular momentum $S^2$.

(b)[5] Rewrite the Hamiltonian in terms of total spin operators $S_z$, and $S^2$.

(c)[4] Show that for a singlet state $S^2 = S_z = 0$, and find the energy.

(d)[4] Find the eigenvalues of $S^2$ and $S_z$ for the triplet states and find the energy.
7. (a)[7] Using the correspondence principle, show that the classical relativistic expression for the energy of a free zero rest mass particle \( E^2 = p^2 c^2 \) transforms into the quantum relativistic wave equation of a spin-\( \frac{1}{2} \) particle

\[
i\hbar \frac{\partial}{\partial t} \Psi = H \Psi = c(\vec{\sigma} \cdot \vec{p}) \Psi
\]

where the components of \( \vec{\sigma} \) are the Pauli matrices and \( \vec{p} = i\hbar \frac{\partial}{\partial x} \) is the linear momentum operator.

(b)[5] Show that the particle angular momentum \( J = \vec{r} \times \vec{p} + \frac{\hbar}{2} \vec{\sigma} \) is conserved.

(c)[5] Without solving the equation, give an argument that the particle spin in a positive (negative) energy state is parallel (antiparallel) to its momentum.
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First Examination — QUANTUM MECHANICS
August 12, 1999

Solve all 6 problems given below.
Each question is worth the same number of points, 17, for a total of 102 points. The numbers in brackets [ ] indicate the number of points each part is worth.
1. (a) [10] Find the energy levels and wave functions of the particle in the one dimensional potential well

\[ U(x) = \begin{cases} 
\alpha \delta(x), & \alpha > 0 \quad |x| < a \\
\infty, & |x| > a 
\end{cases} \]

(b) [7] Assuming that \( \frac{m_0a}{\hbar^2} = \xi >> 1 \), where \( m \) is the mass of the particle, analyze the structure of the low energy part of the spectrum (\( k a << \xi \), where \( k \) is the wave vector of the particle). Demonstrate that the energy spectrum consists of the pairs of closely spaced levels, see figure, and find the energy gap between these closely spaced levels.

Note: A \( \delta \)-like potential \( \alpha \delta(x - x_0) \) in the one-dimensional Schroedinger equation is equivalent to the following boundary conditions on the wave function \( \psi(x) \) at point \( x_0 \):

\[ \psi(x_0 - 0) = \psi_0(x_0 + 0) \]

\[ \psi'(x_0 + 0) - \psi'(x_0 - 0) = \frac{2m_0\alpha}{\hbar^2} \psi(x_0) \]
2.

[17] A particle is placed in the following three-dimensional infinitely deep potential well

\[ U(x, y, z) = \begin{cases} 
0, & \frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} < 1 \\
\infty, & \frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} > 1
\end{cases} \]

Assume that \( a = b(1 + \epsilon), |\epsilon| << 1 \), that is the deviation of the well's shape from the purely spherical well is very small. Using the perturbation theory find the ground state energy of the particle in this well in the first order on small parameter \( \epsilon \).

Hint: Using the scaling transformation reduce the well's shape to a spherical one and then apply the perturbation theory.

Note: The ground state wave function and the energy eigenvalue of the particle in an infinitely deep spherical potential well of radius \( R \) are

\[ \psi_0(r) = \frac{1}{\sqrt{2\pi R}} \sin \frac{\pi r}{R} \quad E_0 = \frac{\pi^2 \hbar^2}{2mR^2} \]
3. 

[17] A particle of charge \( q \) is constrained to move in a plane at a fixed distance \( R \) from some point (charged plane rotator). Find the energy levels and normalized wave functions of this particle in a uniform constant magnetic field perpendicular to the plane.

Hint: Use cylindrical coordinate system and choose the vector potential as \( \vec{A} = \frac{1}{2} \vec{B} \times \vec{r} \) (symmetric gauge).
4.

Starting with the free particle momentum wave function of a gaussian wave packet

$$A(k) = (2\pi)^{-1/4}(2\Delta)^{1/2}e^{-(k-k_0)^2\Delta^2}$$

(a)[5] Show that, in the coordinate representation, it can be written as

$$\Psi(x, t) = (2\pi)^{-1/4}(\Delta + \frac{i\hbar t}{2m\Delta})^{-1/2}\exp\left\{-\frac{x^2}{4\Delta^2} + ik_0x - \frac{(i\hbar t^2)}{2m}\frac{t}{\Delta^2}\right\}$$

(b)[5] Show that

$$|\Psi(x, t)|^2 = (2\pi)^{-1/2}\left[\Delta^2 + \left(\frac{\hbar t}{2m\Delta}\right)^2\right]^{-1/2}\exp\left\{-\frac{(x - \frac{\hbar k_0 t}{m})^2}{2\left[\Delta^2 + \left(\frac{\hbar t}{2m\Delta}\right)^2\right]}\right\}$$

(c)[4] Verify that $\Psi(x, t)$ stays normalized to unity at all times.

(d)[3] Working in the "coordinate representation" calculate $\langle x(t) \rangle$, $\langle p(t) \rangle$ for the above Gaussian wave packet. Is your result for $\langle x(t) \rangle$ consistent with classical result?
5. [17] A particle of mass $m$ and charge $q$ sits in a harmonic oscillator potential $V = \frac{1}{2}k(x^2 + y^2 + z^2)$. At time $t = -\infty$ the oscillator is in its ground state. It is then perturbed by a spatially uniform time-dependent electric field

$$\vec{E}(t) = Ae^{-(t/\tau)^2} \hat{z}$$

($\hat{z}$ is a unit vector in $z$ direction) where $A$ and $\tau$ are constants. Calculate in lowest-order perturbation theory the probability that the oscillator is in an excited state at $t = \infty$. 
6. Consider a spinless particle represented by the wave function

$$\Psi = A[x + y + 2z]e^{-\alpha r}$$

where \( r = \sqrt{x^2 + y^2 + z^2} \), and \( A \) and \( \alpha \) are real constants.

(a) [5] What is the total angular momentum of the particle?

(b) [4] What is the expectation value of the \( z \)-component of angular momentum?

(c) [4] If the \( z \)-component of angular momentum, \( L_z \), were measured, what is the probability that the result would be \( L_z = \hbar \)?

(d) [4] What is the probability of finding the particle at \( \theta \), and \( \phi \) in solid angle of \( d\Omega \)? Here \( \theta \), and \( \phi \) are the usual angles of spherical coordinates.

Useful expressions:

\[
Y_0^0 = \frac{1}{\sqrt{4\pi}} \\
Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta \\
Y_1^{\pm1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \\
Y_2^0 = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1) \\
Y_2^{\pm1} = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi} \\
Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}
\]
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First Examination - QUANTUM MECHANICS

January 13, 2000

Do all of the 6 problems given. Each question is worth the same number of points, 17, for a total of 102 points. The numbers in brackets [ ] indicate the number of points each part is worth.
1. (a) [10] Find transmission and reflection coefficients for a free one-dimensional particle with a wave number $k$ tunneling through a double $\delta$-like potential barrier

\[ U(x) = \alpha \left[ \delta(x) + \delta(x - a) \right], \quad \alpha > 0 \]

(b) [7] At some values of $k$ the double barrier is completely transparent, i.e. the reflection coefficient is equal to zero. Find these values of $k$. (This phenomenon is called resonant tunneling).

Note: A $\delta$-like potential $\alpha \delta(x - x_0)$ in the one-dimensional Schroedinger equation is equivalent to the following boundary conditions on the wave function $\psi(x)$ at point $x_0$:

\[ \psi(x_0 - 0) = \psi(x_0 + 0) \]

\[ \psi'(x_0 + 0) - \psi'(x_0 - 0) = \frac{2m\alpha}{\hbar^2} \psi(x_0) \]
2. The Hamiltonian of a system has the following form

\[ H = \begin{pmatrix} \epsilon_0 & 0 & \alpha \\ 0 & \epsilon_1 & 0 \\ \alpha & 0 & \epsilon_0 \end{pmatrix} \quad \alpha \neq \epsilon_0, \quad \alpha, \epsilon_0, \epsilon_1 \in \mathbb{R} \]

with orthonormal basis states

\[ |1> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

(a) [11] Find the energy eigenvalues and eigenfunctions for this system.

(b) [6] If at \( t = 0 \) the state of the system \( |\psi(0)> = |3> \), what is the probability of finding system in state \( |1> \) at time \( t \)?
Assume that the energy levels are not degenerate.
A particle is in the ground state of an infinitely deep spherical potential well of radius $R$

$$U(r) = \begin{cases} 
0, & r < R \\
\infty, & r > R 
\end{cases}$$

The well is suddenly expanded to the radius $2R$.

(a)[9] Calculate the probability that the particle will be found in the ground state of the expanded well.

(b)[8] Find the energy eigenstate of the expanded well that is most likely to be occupied after the expansion. Note that from the symmetry of the problem it follows that only the states with $\ell = 0$ can be occupied.

You may find useful the following formula for the Laplace operator in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$
4.
A one dimensional harmonic oscillator has mass $m$ and angular frequency $\omega$. A time-dependent state $\psi(t)$ of the oscillator is given at $t = 0$ by

$$\psi(0) = \frac{1}{\sqrt{2s}} \sum |n >$$

where $|n >$ is an eigenstate of the Hamiltonian corresponding to the quantum number $n$, and the summation runs from $n = N - s$ to $n = N + s$, with $N >> s >> 1$.

(a)[10] Show that the expectation value of the displacement varies sinusoidally with amplitude $(\frac{2kN}{\omega})^{1/2}$.

(b)[7] Relate this result to the time variation of the displacement of a classical harmonic oscillator.
5.

The classical angular momentum vector $\vec{L} = \vec{r} \times \vec{p}$ for a particle, becomes a quantum mechanical operator when $\vec{p}$ becomes an operator like $p_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$, etc. and $[x, p_x] = i\hbar$, $[x, p_y] = 0$ etc.

(a) [4] For the quantum operator prove $\vec{L} \times \vec{L} = i\hbar \hat{L}$.

(b) [4] If $H$ is the Hamiltonian of the electron and $[H, L^2] = 0$ prove that eigenstates of $H$ are simultaneous eigenstates of $\vec{L}^2 = \vec{L} \cdot \vec{L}$.

(c) [4] Define $L_+ = L_x + i L_y$, $L_- = L_x - i L_y$ and find the commutation relations for $L_+$, $L_-$ and $L_z$.

(d) [5] Pick a representation for $\vec{L}$ in which the kets are labeled $|\ell m\rangle$, with $\vec{L}^2|\ell m\rangle = \hbar^2 \ell (\ell + 1)$ and $L_z|\ell m\rangle = m\hbar|\ell m\rangle$, find $L_+|\ell m\rangle$, and $L_-|\ell m\rangle$. 
6.
In the Born approximation the scattering amplitude is given by:

\[ f(\theta) = \frac{-m}{2\pi\hbar^2} \int_{\text{all space}} V(\vec{r}) e^{i\vec{\kappa} \cdot \vec{r}} d\tau \]

and the quantity \( \vec{\kappa} \), is the scattering vector defined by \( \vec{\kappa} = \vec{k} - \vec{k}' \), where \( \vec{k} \) is the wave vector of the incident particles, and \( \vec{k}' \) that of the scattered particles.

(a) [5] Show that for a spherically symmetric potential

\[ f(\theta) = \frac{-2m}{\hbar^2 \kappa} \int_0^{\infty} V(r) r \sin \kappa r \ dr \]

(b) [6] Particles are incident on a spherically symmetric potential given by \( V(r) = \frac{\beta}{r} e^{-\gamma r} \) where \( \beta \) and \( \gamma \) are constants. Find the differential scattering cross-section \( \frac{d\sigma}{d\Omega} \).

(c) [6] Use the result from above to derive the Rutherford formula for the scattering of \( \alpha \)-particles, namely, that for \( \alpha \)-particles of energy \( E \) incident on nuclei of atomic number \( Z \). The differential scattering cross section for scattering angle \( \theta \) to the incident direction is

\[ \frac{d\sigma}{d\Omega} = \left( \frac{Ze^2}{8\pi\epsilon_0 E \sin^2 \frac{\theta}{2}} \right)^2 \]
THE CITY UNIVERSITY OF NEW YORK
PH.D. PROGRAM IN PHYSICS

FIRST EXAMINATION IN QUANTUM MECHANICS

Tuesday, January 16, 2001  10 a.m. -- 2 p.m.

INSTRUCTIONS

* This examination consists of seven problems. Solve six problems. Circle below the six(6) problems you wish to have graded for credit.

1  2  3  4  5  6  7

* Begin each problem on a separate page.

* Show all steps and explain your reasoning clearly.
1. A particle of mass \( m \) moves in one dimension according to the Hamiltonian

\[
H_0 = \frac{p^2}{2m} + V(x),
\]

\[
H_0 \psi_n^{(0)}(x) = E_n^{(0)} \psi_n^{(0)}(x).
\]

All eigenfunctions \( \psi_n^{(0)}(x) \) and eigenvalues \( E_n^{(0)} \) are known.

(A) (i) Evaluate \( e^{-\frac{i}{h} P} e^{\frac{i}{h} \lambda x} \), where \( \lambda \) is a constant and \( P \) is the momentum operator.

(ii) Suppose we add a term to the Hamiltonian

\[
H = H_0 + \frac{\lambda}{m} P.
\]

Derive expressions for the eigenvalues and eigenstates of the new Hamiltonian \( H \) in terms of the eigenvalues and eigenstates of \( H_0 \).

(B) (i) By considering \( \frac{\lambda}{m} P \) as a small perturbation, find the first order correction to the energy \( E_n^{(0)} \).

(ii) The second order correction to the energy \( E_n^{(0)} \) can be calculated from the expression

\[
\langle \psi_n^{(0)} | \frac{\lambda}{m} P | \overline{\psi}_n \rangle,
\]

where

\[
| \overline{\psi}_n \rangle = \sum_{j \neq n} | \psi_j^{(0)} \rangle \frac{\langle \psi_j^{(0)} | \frac{\lambda}{m} P | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_j^{(0)}}.
\]

Show that \( | \overline{\psi}_n \rangle \) satisfies the following equation

\[
(H_0 - E_n^{(0)}) | \overline{\psi}_n \rangle + \frac{\lambda}{m} P | \psi_n^{(0)} \rangle = 0.
\]

2. Consider two identical particles of mass \( m \) and spin \( 1/2 \). They interact only through the potential

\[
V(r) = g^2 \left[ \sigma_1 \cdot \sigma_2 - 3 (\sigma_{1z} \sigma_{2z}) \right]
\]

where \( g \) is a real constant and \( \sigma_j \) \( (= \sigma_j \hat{x} + \sigma_j \hat{y} + \sigma_j \hat{z}) \) are Pauli spin matrices which operate on the spin of particle \( j (=1,2) \).
(A) Construct the total spin eigenfunctions for the two-particle states. What is the expectation value of $V$ for each of these states?

(B) Give the energy eigenvalues of all of the bound states.

3. It is possible that those neutrinos associated with electron, muon and tau are different states of a single system, being three orthonormal states called electron-neutrino $\nu_e$, muon-neutrino $\nu_\mu$, and tau-neutrino $\nu_\tau$, and that they are not stationary states. It is also possible that the masses (energy eigenvalues) of this system are not exactly 0. Let $|\psi_i\rangle$ ($i = 1, 2, 3$) be the normalized eigenstates of energy, with eigenvalues $m_i c^2$, and suppose the eigenstates of the "neutrino type" experiment are

$$|\nu_e\rangle = \frac{1}{2} |\psi_1\rangle + \frac{\sqrt{3}}{2} |\psi_2\rangle$$

and

$$|\nu_\mu\rangle = \frac{3}{4} |\psi_1\rangle - \frac{\sqrt{3}}{4} |\psi_2\rangle - \frac{1}{2} |\psi_3\rangle.$$ 

If the system starts at time $t = 0$ in the electron-neutrino state, find the probabilities that at time $t$ it will be found to be in (A) the muon-neutrino state and (B) the tau-neutrino state.

4. Consider the Hamiltonian for a particle in one dimension $H = (p^2/2m) + V(x,t)$, where

$$V(x,t) = \begin{cases} V(x) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \quad V(x) = \begin{cases} V_0 & -a \leq x \leq a \\ 0 & |x| > a \end{cases}.$$ 

(a) Suppose that at $t = 0$ the wave function of the particle is a momentum eigenstate $|k\rangle$, where the momentum eigenstates are normalized so that $\langle k' | k \rangle = \delta(k' - k)$. Find to first order in $V(x,t)$ the amplitude that at any time $t > T$ the particle is in the momentum eigenstate $|k\rangle$.

(b) Now suppose that the momentum space wave function at $t = 0$ is

$$\Psi(k,0) = \frac{\sqrt{2} b}{\pi^{1/4}} e^{-b k^2},$$

where $a \ll b$ and $(\hbar T)/(2mb^2) \ll 1$. Find $\Psi(k,t)$ to first order in $V(x,t)$ for $t > T$.

5. We want to consider the scaling transformation $\psi(r) \rightarrow N(\lambda) \psi(\lambda r)$.

(a) If $\psi(r)$ is normalized, find $N(\lambda)$ so that $N(\lambda) \psi(\lambda r)$ is also normalized.

(b) In the theory of both Bose-Einstein condensates and polarons, the energy expression
\[ E = \frac{\hbar^2}{2m} \int d^3r |\nabla \psi|^2 - g \int d^3r |\psi|^4 , \]

where \( g > 0 \) occurs. Show, by using a scaled wave function, that this energy has no lower bound.

c) Now consider the energy expression

\[ E = \frac{\hbar^2}{2m} \int d^3r |\nabla \psi|^2 + \int d^3r V(\mathbf{r})|\psi|^2 , \]

where \( V(\mathbf{r}) = a r^2 + b r^4 \). Now let \( \psi(\mathbf{r}) = N(\lambda)\psi(\lambda r) \), and assume that \( \psi(\mathbf{r}) \) is the ground state of the system. This gives us an energy \( E(\lambda) \), that depends on \( \lambda \). Because \( \psi(\mathbf{r}) \) is the ground state, \( E(\lambda) \) is a minimum when \( \lambda = 1 \). Use this fact to derive a relation between the kinetic energy, \( \langle r^2 \rangle \), and \( \langle r^4 \rangle \) in the ground state.

6. a) Use the Born approximation to find the differential cross section for a nonrelativistic particle of mass \( m \) scattering off the potential \( V(\mathbf{r}) = -V_0 e^{-\alpha r} \), where \( V_0 > 0 \).

b) Find the total cross section from your answer in part (a).

7. Consider the two Dirac spinors

\[
\begin{pmatrix}
1 \\
0 \\
p_x \\
p_x + ip_y \\
E + mc^2 \\
E + mc^2
\end{pmatrix}
\quad
\begin{pmatrix}
p_z \\
E + mc^2 \\
p_x + ip_y \\
E + mc^2 \\
1 \\
0
\end{pmatrix}
\]

The wave function \( u(\mathbf{p})e^{ipr} \) is a positive energy solution with momentum \( \mathbf{p} \) and \( v(\mathbf{p})e^{-ipr} \) is a negative energy solution with momentum \( \mathbf{p} \).

a) If the electron is in the state

\[ \psi(\mathbf{r}) = \int dp^3 b(\mathbf{p})u(\mathbf{p})e^{ipr} , \]

find the expectation value of its velocity in terms of its expansion coefficients \( b(\mathbf{p}) \).

b) Consider the wave function
\[ \psi(r) = \frac{1}{(\pi d^2)^{3/4}} e^{-\frac{1}{2} r^2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

Find the condition for \( d \) so that when it is satisfied, the inner product of \( \psi(r) \) with the positive energy solution \( u(p) e^{ipr} \) is comparable in size to its inner product with the negative energy solution \( v(p) e^{-ipr} \).
First Examination — QUANTUM MECHANICS  
August 22, 2002

Do any 5 out of 6 problems. You must indicate which problems you want counted — if you do not do so then problems 1–5 will be graded. Each question is worth the same number of points, 20, for a total of 100 points. The numbers in brackets [ ] indicate the number of points each part is worth.
1. Consider the one dimensional problem where the potential is given by

\[ V(x) = -Fx \]

and where \( F \) is a constant.

a) [5] Write and simplify the Heisenberg equation of motion for the position and momentum operators.

b) [5] Solve these equations for the position and momentum operators as functions of time.

c) [5] Find the expectation values of position and momentum, and their standard deviations (uncertainties) as functions of time.

d) [5] Suppose the position wave function at time zero is given by

\[ \psi(x, 0) = e^{-\alpha x^2 + ikx} \]

where \( \alpha \) and \( k \) are constants and \( \alpha \) is greater than zero. Using your results from part c) find the expected values of position, momentum, and their standard deviations as functions of time.

2. Consider a particle in a spherically symmetric potential

\[ V(x) = kr \]

where \( k \) is positive.

a) [10] Use the variational trial function \( e^{-r} \) to estimate the ground state energy.

You will need the following integral

\[ \int_0^\infty r^n e^{-\beta r} \, dr = \frac{n!}{\beta^{n+1}}. \]

b) [10] Use the uncertainty principle to estimate the ground state energy

3. Consider the one dimensional problem where the potential is given by

\[ V(x) = \begin{cases} \gamma x & x > 0 \\ \infty & x \leq 0 \end{cases} \]

(1)

(2)

where \( \gamma \) is greater than zero.

a) [5] Write Schrödinger’s time independent equation for this problem. Make the transformation

\[ x' = ax + b \]

where \( a, b \) are arbitrary numbers.

b) [6] Find, \( a, b \) so that the transformed Schrödinger equation has the form

\[ \frac{d^2}{dx'^2} u(x') - x'u(x') = 0 \]

c) [8] Assume you know \( u(x) \) and also assume that you know the roots \( r_1, r_2 \ldots \)

\[ u(r) = 0 \quad r = r_1, r_2 \ldots \]

Write the eigenfunctions and eigenvalue of energy of the original Schrödinger equation in terms of \( u(r) \) and its roots. Your answer should be explicit in terms of \( m, \hbar, \gamma, r_n \).
4. The free Dirac Hamiltonian for a particle of mass \( m \), momentum, \( \mathbf{p} \) and spin \( \frac{1}{2} \) can be written as

\[
H = c \alpha \cdot \mathbf{p} + \beta mc^2
\]

where

\[
\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\sigma = \sigma_x e_x + \sigma_y e_y + \sigma_z e_z
\]

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

and \( c \) is the speed of light. The helicity operator is defined by

\[
h(p) = \begin{pmatrix} \sigma \cdot \mathbf{p} / |\mathbf{p}| & 0 \\ 0 & \sigma \cdot \mathbf{p} / |\mathbf{p}| \end{pmatrix}.
\]

(A) [4] Evaluate the commutator \([H, h(p)]\).

(B) [8] Calculate \( h^2(p) \) and show that the eigenvalues of \( h(p) \) are \( \pm 1 \).

(C) [8] Define

\[
w(p, \lambda = +1) = \begin{pmatrix} \phi_+ \sqrt{|\mathbf{p}|} \\ \phi_+ \sqrt{E + mc^2} \end{pmatrix}
\]

and

\[
w(p, \lambda = -1) = \begin{pmatrix} \phi_- \sqrt{|\mathbf{p}|} \\ \phi_- \sqrt{E + mc^2} \end{pmatrix}
\]

where \( \phi_+ \) and \( \phi_- \) are the two-component helicity spinors. \( w(p, \lambda = +1) \) and \( w(p, \lambda = -1) \) will be the helicity eigenstates (with eigenvalues \( +1 \) and \( -1 \), respectively),

\[
\begin{align*}
h(p)w(p, \lambda = +1) &= +w(p, \lambda = +1) \\
h(p)w(p, \lambda = -1) &= -w(p, \lambda = -1)
\end{align*}
\]

if the helicity spinors \( \phi_+ \) and \( \phi_- \) satisfy certain conditions. Find these conditions.

(You may assume without proof \( \sigma \cdot \mathbf{A} \sigma \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B} + i\sigma \cdot (\mathbf{A} \times \mathbf{B}) \))
5. We consider the scattering of a particle of mass $m$, momentum $hk$ and angular momentum $\ell$ by a potential $V(r)$. The scattering amplitude $f(\theta)$ can be written in terms of the phase shift $\delta_\ell(k)$ as

$$f(\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell + 1) e^{i\delta_\ell} \sin \delta_\ell P_\ell(\cos \theta).$$

where $P_\ell$ and $\theta$ are Legendre polynomials and the scattering angle, respectively. The differential cross section $d\sigma/d\Omega$ and the total cross section $\sigma$ can be calculated from this amplitude.

(A) [8] Prove the optical theorem which relates the imaginary part of the forward scattering amplitude, $\text{Im} f(\theta = 0)$, to the total cross section $\sigma$ as follows:

$$\text{Im} f(\theta = 0) = \frac{k}{4\pi} \sigma$$

$$= \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell + 1) e^{i\delta_\ell} \sin^2 \delta_\ell$$

(B) [12] Consider the special case of a hard sphere potential

$$V(r) = \begin{cases} \infty & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

(i) Derive the expression for $\delta_\ell$ in terms of the spherical Bessel function $j_\ell(ka)$ and the spherical Neumann function $n_\ell(ka)$

(ii) Find the total cross section $\sigma$.

(iii) Show that the low-energy scattering is always dominated by the s-wave ($\ell = 0$). (When $k \to 0$, $j_\ell(ka) \approx (ka)^{\ell}/(2\ell + 1)$! and $n_\ell(ka) \approx -(2\ell - 1)!!/(ka)^{\ell+1}$ where $(2\ell - 1)!! = 1$ for $\ell = 0$.

(iv) What is the total cross section $\sigma$ in the extreme low-energy limit $k \to 0$? Compare your answer with the geometric cross section $\pi a^2$.

$$j_0(x) = \sin x / x ; \quad n_0(x) = -\cos x / x$$

6. Consider a system composed of two spin-\(\frac{1}{2}\) particles. The Hamiltonian of the system is given by

$$H = 4A s_{1z} s_{2z} + B(s_{1z} + s_{2z})$$

where $A$ and $B$ are two real constants and $s_{1z}(s_{2z})$ is the $z$-component of the spin $s_1(s_2)$ of particle 1 (particle 2).

(A) [6] Determine the eigenvalues and eigenvectors of the Hamiltonian operator $H$.

(B) [14] Now, consider the case in which the Hamiltonian of the system vanishes (zero) for $t < 0$, and, for $t > 0$, is given by $H$. Suppose the system is in the state

$$|\Psi(t \leq 0)\rangle = \frac{1}{\sqrt{2}} |1\rangle - \frac{1}{\sqrt{2}} |2\rangle \equiv |\beta_1\alpha_2\rangle$$

for $t \leq 0$. ($|\beta_1\alpha_2\rangle$ is an eigenstate of $s_{1z}$ and $s_{2z}$ with the eigenvalues $-\frac{3}{2}$ and $+\frac{1}{2}$, respectively.) Calculate the probability of finding the system, at time $t$, in each of the following states $|\alpha_1\alpha_2\rangle$, $|\alpha_1\beta_2\rangle$, $|\beta_1\alpha_2\rangle$ and $|\beta_1\beta_2\rangle$ using (i) the exact solution (without approximation) and (ii) the first-order time-dependent perturbation theory with $H$ as a perturbation switched on at $t = 0$. 

©
Do five of the following six problems. Start each problem on a new page. Indicate clearly which five problems you choose to solve. If you do not indicate which problems you wish to be graded, the first five problems will be graded. PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.

1) (20 points) A one-dimensional Gaussian Wave packet

\[ \psi(x,0) = \frac{1}{\pi^{1/4}(\Delta x)^{1/2}} \exp \left( \frac{i p_0 x}{\hbar} - \frac{x^2}{2(\Delta x)^2} \right) \]

describes a free electron confined at time \( t = 0 \) within a distance \( \Delta x = 10^{-10} \text{m} \). Estimate the size of the wave packet after a time \( t \sim 10^{-16} \text{s} \).

2) (20 points) Consider the motion of a particle of mass \( m \) in a one-dimensional potential well \( V(x) \). Use the WKB approximation to find all the bound state energy levels for the potential

\[ \begin{align*}
V(x) &= 0 \quad \text{for} \quad |x| > a \\
V(x) &= -V_0 \left( 1 - \frac{|x|}{a} \right) \quad \text{for} \quad |x| < a
\end{align*} \]

where \( V_0 > 0 \).

3) a) (10 points) A quantum-mechanical system is initially in state \( |i \rangle \) of the unperturbed Hamiltonian \( H^0 \). At time \( t = 0 \), a small perturbation \( H^1 \) is applied to the system. Determine the probability that the system is in a different state \( |f \rangle \) of the unperturbed Hamiltonian at time \( t > 0 \).

b) (10 points) Now consider the one-dimensional Hamiltonian

\[ H_0 = \frac{p^2}{2m} + \frac{1}{2} kx^2 \]

for a particle of mass \( m \) and charge \( q \). At time \( t = 0 \) a homogeneous electric field \( E(t) = E_0 \exp(-t/\tau) \) (\( E_0 \) and \( \tau \) are positive constants) directed along the x-axis is switched on. If the oscillator is in the ground state of the unperturbed system for \( t \leq 0 \), find the probability that it is in a different state \( f \) as \( t \to \infty \). You may use the fact that for \( \omega = \sqrt{k/m} \), the creation and annihilation operators for the harmonic oscillator are given by

\[ \begin{align*}
a^\dagger &= \left( \frac{m\omega}{2\hbar} \right)^{1/2} x + i \left( \frac{1}{2m\omega\hbar} \right)^{1/2} p \\
a &= \left( \frac{m\omega}{2\hbar} \right)^{1/2} x - i \left( \frac{1}{2m\omega\hbar} \right)^{1/2} p
\end{align*} \]

\[ \begin{align*}
a^\dagger |n \rangle &= \sqrt{n+1} |n+1 \rangle \\
a |n \rangle &= \sqrt{n} |n-1 \rangle
\]
4) (20 points) Determine the energy spectrum of a charged particle moving in the background of a uniform electric field along the x-axis and a uniform magnetic field along the z-axis.

5) a) (10 points) Starting with the Schrodinger equation, and using the Green Function approach, derive the expression for the scattering amplitude in the Born approximation. You may use the fact that the Green function solving

\[
(\nabla^2 + k^2) G^0(\vec{r}, \vec{r}') = \delta^3(\vec{r} - \vec{r}')
\]

is given by

\[
G^0(\vec{r}, \vec{r}') = G^0(\vec{r} - \vec{r}') = -\frac{e^{ik|\vec{r} - \vec{r}'|}}{4\pi|\vec{r} - \vec{r}'|}
\]

b) (10 points) Determine the scattering amplitude, differential scattering cross section and total scattering cross section for the potential

\[
V(r) = V_0 \frac{a^2}{r^2 + a^2}
\]

in the Born approximation, where \(a^2\) is a positive constant.

6) (20 points) The Dirac-theory Hydrogen Atom Hamiltonian is given by

\[
H = c \vec{\alpha} \cdot \vec{p} + \beta mc^2 - \frac{e^2}{r}
\]

Calculate the time derivative of

\[
K = \beta \left( \frac{\vec{L} \cdot \vec{s}'}{\hbar^2} + \frac{1}{2} \right)
\]

where \(\vec{L}\) is the orbital angular momentum operator and \(\vec{s}' = \frac{\hbar}{2} \vec{S}'\) is the spin operator. The four-dimensional matrices \(\vec{\alpha}, \vec{\beta}\) and \(\vec{\Sigma}\) are given by

\[
\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \vec{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}
\]

and where \(\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)\) and the \(\sigma_i\) are the Pauli matrices, which satisfy

\[
[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k
\]
THE CITY UNIVERSITY OF NEW YORK
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics
January 22, 2009

Instructions:
(1) Put your identification number on each page.
(2) Solve two of the following three problems. Each problem is worth 25 points.
(3) Start each problem on a new page.
(4) Indicate clearly which two problems you choose to solve. If you do not indicate which problems you
wish to be graded, the first two problems will be graded.

1. Heisenberg’s equation of motion for an operator \( A \) that is not explicitly time dependent is

\[
\frac{dA}{dt} = \frac{1}{i\hbar} [A, H] \tag{1}
\]

and \( H = \frac{p^2}{2m} + V(q) \). Assume that Taylor’s series holds for operators

\[
A(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n}{dt^n} A(t) \right|_{t=0} t^n \tag{2}
\]

(a) [9] Define \( C_H \) operating on an operator \( A \) by

\[
C_H A = [A, H] \tag{3}
\]

and also \( C^n_H A = C_H C_H^{n-1} A \) (For example \( C_H^2 A = C_H C_H A = C_H [A, H] = [[A, H], H] \) ) Show that

\[
A(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{t}{i\hbar} \right)^n C^n_H A \bigg|_{t=0} \tag{4}
\]

(b) [8] Consider the Hamiltonian \( H = \frac{p^2}{2m} - Fq \) where \( F \) is a constant. Find \( C^n_H A \big|_{t=0} \) for all

\( n \) for the case \( A = q \). Hence write \( q(t) \) as given by Eq. (4).

(c) [8] Obtain \( q^2(t) \) by squaring the answer you got in part b). Define

\[
\sigma^2(t) = \langle q^2(t) \rangle - \langle q(t) \rangle^2 \tag{5}
\]

and obtain an explicit expression for \( \sigma^2(t) \) in powers of \( t \).

2. Consider the one dimensional problem where the potential is given by

\[
V(x) = \begin{cases} 
-\frac{A}{x^2} + \frac{B}{x}, & x > 0 \\
\infty & x \leq 0
\end{cases} \tag{6}
\]

(a) [12] Use the variational principle to estimate the ground state energy and the ground state wave
function. Use the following trial wave function

\[
\psi(x) = N x e^{-\alpha x}
\]

where \( \alpha \) is the variational parameter.
(b) [7] After you have obtained your answer consider the special case where \( B = 0 \). By setting \( B = 0 \) for what you obtained in part a) calculate \( H\psi(x) \) to show that it is an exact eigenstate.

(c) [6] Using the results of part b) calculate the correction to the \( \text{energy} \) for the case of the potential given by Eq. (5) by using first order stationary state perturbation theory. Compare your answer for the energy with that obtained in part a) and comment.

3. Parts A and B for this question are independent of each other

(a) PART A:

(b) [8] Consider the following three dimensional wave function

\[
\psi(r, \theta, \varphi) = N \left[(i - 1)Y_1^1(\theta, \varphi) + 3Y_1^0(\theta, \varphi) + (i + 2)Y_1^{-1}(\theta, \varphi)\right] re^{-\alpha r} \tag{7}
\]

Normalize the wave function and calculate the probabilities of measuring \( m = 0, 1, -1 \).

(c) [7] Suppose you are told that \( \psi(r, \theta, \varphi) \) is an eigenfunction of some central potential \( V(r) \) and that the energy eigenvalue is \( E \). What is that potential?

(d) [10] PART B: The aim of this part is to prove the Ehrenfest’s theorem for angular momentum/torques: for a particle in a potential \( V(x, y, z) \), the rate of change of the expectation value of the orbital angular momentum \( \overrightarrow{L} \) is equal to the expectation value of the torque, defined by \( (\overrightarrow{r} \times (-\nabla V)) \): That is,

\[
\frac{d}{dt} < \overrightarrow{L} > = < \overrightarrow{r} \times (-\nabla V) > \tag{8}
\]

Prove this any way you want but the easiest is to consider

\[
\frac{d}{dt} < \overrightarrow{L} > = \frac{1}{i\hbar} < [L, H] > \tag{9}
\]

and to do it for one component, say \( L_z \).

Important: Take

\[
H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z) \tag{10}
\]

and \( V \) is an arbitrary potential – not necessarily a central force potential. You must show you work in detail and do not guess at any answers.
\[ \int_0^\infty e^{-sx}x^n dx = \frac{n!}{s^{n+1}}. \] (1)

\[ L_x = \frac{\hbar}{i}(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}) \] (2)

\[ L_y = \frac{\hbar}{i}(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}) \] (3)

\[ L_z = \frac{\hbar}{i}(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}) \] (4)

\[ \vec{a} \times \vec{b} = (a_2b_3 - a_3b_2) \hat{i} + (a_3b_1 - a_1b_3) \hat{j} + (a_1b_2 - a_2b_1) \hat{k} \] (5)
THE CITY UNIVERSITY OF NEW YORK
First Examination for PhD Candidates in Physics – Quantum Mechanics
June 16, 2009

Instructions:
a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.
b) Solve two of the following three problems. Each problem is worth 25 points.
c) Start each problem on a new page.
d) Indicate clearly which two problems you choose to solve. If you do not indicate which problems you wish to be graded, the first two problems will be graded.

1. (a) [5] Consider the one dimensional problem where the potential is

\[
V(x) = \begin{cases} 
V_0 - a\delta(x), & \text{if } x \geq 0 \\
-a\delta(x) & \text{if } x \leq 0
\end{cases}
\] (1)

and where \( a \) is a positive constant. That is, we have a delta function potential over all space given by \( -a\delta(x) \) and in addition we have a constant potential only to the right of \( x = 0 \). Using Schrodinger’s time independent equation show that the derivative of the wave function has a discontinuity at \( x = 0 \):

\[
\left. \frac{du}{dx} \right|_{0^+} - \left. \frac{du}{dx} \right|_{0^-} = -a \frac{2m}{\hbar^2} u(0)
\] (2)

(b) [20] Find the reflection and transmission coefficients. Assume that the incoming particles are coming from minus infinity and traveling to the right. Consider the case where the energy is greater than \( V_0 \).

2. (a) [5] Show that

\[
\delta(x) = \frac{1}{2} \frac{d^2}{dx^2} |x|
\]

(b) [20] Use the following wave trial wave function

\[
\psi(x) = Ae^{-c|x|/2}
\] (3)

to estimate the ground state energy of the harmonic oscillator where

\[
H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}
\] (4)

Note that \( x \) goes from \(-\infty\) to \(+\infty\)

3. (a) [17] A particle in three dimensional is constrained to be within a spherical shell of inner radius \( R_1 \) and outer radius \( R_2 \). Assume that the wave function vanishes at \( R_1 \) and \( R_2 \). Find the energy eigenfunctions and eigenvalues for the states where \( l = 0 \). Make sure you normalize the eigenfunctions. [Hint: to make the algebra come out easier consider taking \( \psi(r) = A \frac{\sin(k(r-R_1))}{r} \) but you do not have to do that if you have your own way.]

(b) [8] Find \( \langle r \rangle \) and the standard deviation of \( r \) for an arbitrary state that you obtained in part a)
Equations

\[ \int_0^\infty x^n e^{-sx} \, dx = \frac{n!}{s^{n+1}} \]  \hspace{1cm} (5)

\[ \int_0^\pi \sin^2 nx \, dx = \frac{\pi}{2} \]
\[ \int_0^\pi x \sin^2 nx \, dx = \frac{\pi^2}{4} \]
\[ \int_0^\pi x^2 \sin^2 nx \, dx = \frac{\pi^3}{6} - \frac{\pi}{4n^2} \]
1. (a) [3] For a harmonic oscillator potential
\[ H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \]
suppose the wave function at \( t = 0 \) is given by
\[ \psi(x, 0) = A \sum_{n=0}^{\infty} a_n u_n(x) \]
where \( A \) is a normalizing constant, \( a_n \) are given constants, and \( u_n(x) \) are the harmonic oscillator eigenfunctions. Find \( A \) and write \( \psi(x, t) \).

(b) [6] Show that the expected value of position is given by
\[ \langle x \rangle_t = \frac{\alpha A^2}{\sqrt{\pi}} \sum_{n=0}^{\infty} a_n \left[ \frac{n+1}{2} a_{n+1} e^{i\omega t} + \sqrt{\frac{n}{2}} a_{n-1} e^{-i\omega t} \right] \]
and obtain a similar expression for \( \langle p \rangle_t \). Also find \( \langle H \rangle_t \) where \( H \) is the Hamiltonian.

(c) [4] Heisenberg’s equation of motion for the expectation value of an operator is \( \frac{d}{dt} \langle O \rangle_t = \frac{\hbar}{i} \langle [O, H] \rangle_t \). Show that Eq. (1) and your solution for \( \langle p \rangle_t \) satisfies Heisenberg’s equation for \( \langle x \rangle_t \) and \( \langle p \rangle_t \).

(d) [8] Consider the following case:
\[ a_N = a_{N+1} = \ldots = a_{N+K-1} = 1 \]
for some \( N \) and \( K \) and all other \( a_n \) are zero. Also assume \( N >> K >> 1 \). Show that
\[ \langle x \rangle_t = a \cos \omega t \]
and find the constant \( a \). Also find \( \langle p \rangle_t \). Comment on your answers.

(e) [4] Consider the wave function given by
\[ \psi(x, 0) = x^2 e^{-\alpha^2 x^2 / 2} \]
where as usual \( \alpha = \sqrt{\frac{m\omega}{\hbar}} \). Put \( \psi(x, 0) \) in the form given by Eq. (1) Find the coefficients \( a_n \) and the normalizing constant \( A \).
2. (a) [8] For a beam of particles traveling in the z direction and that are scattered by a short range symmetrical potential centered at the origin the main idea is to write the total wave function as

\[ \psi(r) = e^{ikz} + \frac{f(\theta)}{r} e^{ikr} \]

where \( z = r \cos \theta \) and where \( f(\theta) \) is the scattering amplitude. Write explicitly in the appropriate coordinate system the expression for the total current corresponding to the total wave function, \( \psi(r) \). Do not try to simplify it. From the expression you wrote pick out the current for the incoming wave, \( e^{ikz} \), and the current in the radial direction for the outgoing wave \( \frac{f(\theta)}{r} e^{ikr} \). Also, obtain the relationship between the outgoing and incoming current.

(b) [9] The Born approximation is that

\[ f(\theta) = -\frac{m}{2\pi \hbar^2} \int V(r) e^{i(k-k') \cdot r} d^3r \]

where \( k' \) and \( k \) are the momentum of the incoming and outgoing particles for a spherically symmetric potential. Show that

\[ f(\theta) = -\frac{2m}{\hbar^2 K} \int_0^\infty rV(r) \sin(Kr) dr \]

where \( K = |K| = 2k \sin(\theta/2) \)

(c) [8] Suppose the potential is

\[ V(r) = \eta e^{-\alpha r} \]

where \( \eta \) is a constant. Calculate \( f(\theta) \).

3. Parts a, b, c are independent of each other.

(a) [8] A harmonic oscillator is in the ground state. This spring constant is suddenly increased by a factor of \( c^4 \). That is \( k_{\text{new}} = c^4 k_{\text{old}} \). Find the probability that it will be in the ground state of the new Hamiltonian and also the probability that it will be in an excited state. Also, find what happens in the limits as \( c \to 0 \) and \( c \to \infty \).

(b) [10] For the potential

\[ V(x) = \begin{cases} \\
\frac{kx^2}{2} & x > 0 \\
\infty & x \leq 0 
\end{cases} \]

use the trial function

\[ \psi(x) \sim xe^{-\alpha x/2} \quad x > 0 \]

(where \( \alpha \) is the variational parameter) to estimate the ground state energy.

(c) [7] Suppose the Hydrogen atom potential is replaced by the following potential

\[ V(r) = \begin{cases} \\
-\frac{e^2}{r} & r \leq r_0 \\
-\frac{e^2}{r} e^{-\eta(r-r_0)} & r > r_0 
\end{cases} \]

where \( \eta \) is a small positive number. We want to use stationary state perturbation theory to find the correction to the ground state energy. To do so note that the perturbation can be taken to be

\[ H'(r) = \frac{e^2}{r} \left( 1 - e^{-\eta(r-r_0)} \right) \quad r > r_0 \]

Use stationary state perturbation theory to get the first order correction to the ground state.
Equations

$\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$

quantum current $= \frac{\hbar}{2mi} (\Psi^* \nabla \Psi - \Psi \nabla \Psi*) = \frac{\hbar}{m} \text{Im}(\Psi^* \nabla \Psi) = \text{Re}(\Psi^* \frac{\hbar}{im} \nabla \Psi)$

$\int_{-\infty}^{\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$

$\int_{-\infty}^{\infty} xe^{-ax^2+bx} = \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/4a}$

$\int_{-\infty}^{\infty} x^2e^{-ax^2+bx} = \frac{2a+b^2}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/4a}$

$\int_0^{\infty} x^n e^{-sx} dx = \frac{n!}{s^{n+1}}$

$\int_{-\infty}^{\infty} x e^{-sx} dx = \frac{sR+1}{s^2} e^{-sR}$

$\int_{-\infty}^{\infty} e^{-sx} dx = \frac{1}{s} e^{-sR}$

*************************************************************

Hydrogen atom:

$a_0 = \frac{\hbar^2}{me^2}$

$E_n = -\frac{Z^2e^2}{2a_0 n^2} = -\frac{Z^2me^4}{2\hbar^2 n^2}$

Ground state wave function: $\frac{1}{\sqrt{\pi}} \left( \frac{1}{a} \right)^{3/2} e^{-r/a}$

Ground state energy: $-\frac{e^2}{2a} = -\frac{me^4}{2\hbar^2}$

 Harmonic oscillator:

$u_n(x) = N_n e^{-\alpha^2 x^2/2} H_n(\alpha x)$

$u_n(x) = \left( \frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} e^{-\alpha^2 x^2/2} H_n(\alpha x)$

$E_n = (n + 1/2) \hbar \omega$ \hspace{1cm} $\alpha = \sqrt{\frac{m \omega \hbar}{2}} = \left( \frac{mk}{\hbar^2} \right)^{1/4}$

$\omega = \sqrt{\frac{k}{m}}$ \hspace{1cm} $N_n = \left( \frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2}$

$H_0(x) = 1$ \hspace{1cm} $H_1(x) = 2x$ \hspace{1cm} $H_2(x) = 4x^2 - 2$

$u_0(x) = \left( \frac{\alpha^2}{\pi} \right)^{1/4} e^{-\alpha^2 x^2/2}$ \hspace{1cm} $u_1(x) = \sqrt{2} \alpha \left( \frac{\alpha^2}{\pi} \right)^{1/4} e^{-\alpha^2 x^2/2}$

$\int u_n(x) u_m(x) = \frac{1}{\alpha} \left[ \sqrt{\frac{n+1}{2}} \delta_{m,n+1} + \sqrt{\frac{n}{2}} \delta_{m,n-1} \right]$ \hspace{1cm} $\int u_n(x) p u_m(x) = i\alpha h \left[ \sqrt{\frac{n+1}{2}} \delta_{m,n+1} - \sqrt{\frac{n}{2}} \delta_{m,n-1} \right]$

*************************************************************

Hydrogen atom:

$u_0(x) = \frac{\hbar^2}{me^2}$

$E_n = -\frac{Z^2e^2}{2a_0 n^2} = -\frac{Z^2me^4}{2\hbar^2 n^2}$

Ground state wave function: $\frac{1}{\sqrt{\pi}} \left( \frac{1}{a} \right)^{3/2} e^{-r/a}$

Ground state energy: $-\frac{e^2}{2a} = -\frac{me^4}{2\hbar^2}$
THE CITY UNIVERSITY OF NEW YORK
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics
June 17, 2010

Instructions:

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PART A

1. Parts a, b, and c are independent of each other but you may use the results of part a to do parts b and c

   (a) [5] If the wave function at $t = 0$ is given by $\psi(x, 0)$ show that
   
   $$\psi(x, t) = e^{-iHt/\hbar}\psi(x, 0)$$
   
   satisfies use Schrodinger time dependent equation where $H$ is a time independent Hamiltonian. Also by expanding $\psi(x, 0)$ in terms of the the eigenfunctions of the Hamiltonian $\psi(x, 0) = \sum_{n=0}^{\infty} c_n u_n(x)$ show that
   
   $$\psi(x, t) = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} u_n(x)$$
   
   where $u_n$ and $E_n$ are the eigenfunctions and eigenvalues of the Hamiltonian.

   (b) [13] Suppose we are dealing with a harmonic oscillator where and we have the following wave function at time zero
   
   $$\psi(x, 0) = \left(\frac{m\omega}{\hbar \pi}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}(x-\beta)^2}$$
   
   Find $\psi(x, t)$. Hint: calculate $c_n$ and substitute in Eq. () and then do the infinite summation using the generating function which is given in the equation sheet. In calculating $c_n$ a difficult integral that arises but the answer is given in the formula sheet.

   (c) [7] As in part a)
   
   $$\psi(x, t) = e^{-iHt/\hbar}\psi(x, 0)$$
   
   Assume that $t$ is small and expand $e^{-iHt/\hbar}$ to second order in time. Then, calculate
   
   $$P(t) = \left|\int \psi^*(x, 0)\psi(x, t) dx\right|^2 = P(t)$$
   
   which is called the survival probability. Find $P(t)$ to second order in time. That is, find $A$, $B$, $C$ where $P(t) \sim A + Bt + Ct^2$. [The coefficients are constants and what enters in $A,B,C$ are either numerical values or expectation values of the Hamiltonian at time
2. Parts d and e are independent of a,b,c

(a) [5] Consider the delta function potential

\[ V(x) = -\lambda \delta(x) \]

Find the bound states and corresponding energies. Hint try: \( u(x) = Ae^{-a|x|/2} \). Also you will need the fact that the discontinuity of the eigenfunctions satisfy

\[ \left. \frac{du}{dx} \right|_{0^+} - \left. \frac{du}{dx} \right|_{0^-} = -\lambda \frac{2m}{\hbar^2} u(0) \]

You should prove this if you use it.

(b) [5] Calculate \( \langle x \rangle \), \( \langle p \rangle \), \( \langle x^2 \rangle \), \( \langle p^2 \rangle \) and the uncertainty product \( \Delta x \Delta p \) for the solution you obtain. You may use the fact that

\[ \delta(x) = \frac{1}{2} \frac{d^2}{dx^2} |x| \]

(c) [5] Now consider the two dimensional potential

\[ V(x, y) = -\lambda_1 \delta(x) - \lambda_2 \delta(y) \]

Use separation of variables to find the energy eigenvalues and eigenfunctions.

(d) [5] If we have a Hamiltonian that depends on a parameter, say \( \theta \); then when one solves the eigenvalue problem

\[ H(x, p; \theta) u(x; \theta) = E(\theta) u(x; \theta) \]

the energy eigenvalues ad eigenfunctions will contain the parameter. Show that

\[ \frac{\partial E}{\partial \theta} = \int u^*(x; \theta) \left( \frac{\partial H}{\partial \theta} \right) u(x; \theta) dx \]

where \( u(x; \theta) \) is first normalized to one. This is called the Hellmann -Feynman theorem.

(e) [5] Consider the harmonic oscillator Hamiltonian

\[ H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} kx^2 \]

and take the parameter to be the mass \( m \). Calculate \( \int u^*(x; \theta) \left( \frac{\partial H}{\partial m} \right) u(x; \theta) dx \) for the ground state of the harmonic oscillator. Since we know the exact answer for the ground state check your answer by calculating \( \frac{\partial E_0}{\partial m} \) where \( E_0 \) is the ground state energy.
PART B

3. A relativistic electron moves in one dimension along the z-axis and interacts with the potential

\[ V(x) = \frac{m\omega^2 z^2}{4} (I + \beta) \]

where \( I \) is a 4x4 unit matrix and \( \beta \) is a standard Dirac matrix. (Note that the electron is 3-dimensional, but its wavefunction only depends on the z-direction).

(a) [3] Write down the Dirac equation governing the motion.
(b) [3] What is the physical meaning of the quantity \( \psi^+ \psi \)
(c) [6] Write the four-component wave function in the form

\[ \psi = \begin{pmatrix} u \\ v \end{pmatrix} \]

where \( u \) and \( v \) are two-component wave functions. Solve for \( v \) in terms of \( u \).
(d) [7] Find \( \psi \) for the ground state assuming that the electron is in a spin-up state along the z-axis.
(e) [3] Find an algebraic formula to determine the energy eigenvalue \( E \). You need not solve for \( E \).
(f) [3] Solve for the energy in the non-relativistic limit and comment on its value.

4. A particle of mass \( m \) travels with wave vector of magnitude \( k \) along the z-direction and scatters off the potential given by the spherical well

\[ V(\vec{r}) = \begin{cases} -V_0 & \text{if } r < a \\ 0 & \text{if } r > a \end{cases} \]

(a) [8] Find the scattering amplitude \( f(q) \) in the first Born approximation, where \( q \) is the magnitude of the wave-vector transfer.
(b) [4] Find the corresponding differential scattering cross section \( \frac{d\sigma}{d\Omega} \)
(c) [7] Next consider a different scattering potential given by a cubic well

\[ V(x, y, z) = \begin{cases} -V_0 & \text{if } |x| < a \text{ and } |y| < a \text{ and } |z| < a \\ 0 & \text{otherwise} \end{cases} \]

Find the scattering amplitude \( f(q, \theta, \varphi) \), where the wave-vector transfer \( \vec{q} \) is expressed in its spherical coordinate form.
(d) [6] Find the corresponding differential scattering cross section \( \frac{d\sigma}{d\Omega} \). Compare the scattering cross sections for parts b and d in the limit where \( qa << 1 \).
Formulas

\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2 \right) u_n(x) = E_n u_n(x)
\]  

(1)

\[ u_0(x) = \left( \frac{\alpha}{\sqrt{\pi}} \right)^{1/2} e^{-\alpha^2 x^2/2} \]

(2)

\[ u_n(x) = N_n e^{-\alpha^2 x^2/2} H_n(\alpha x) \]

(3)

\[ u_0(x) = \left( \frac{\alpha}{\sqrt{\pi}} \right)^{1/2} e^{-\alpha^2 x^2/2} \]

(4)

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega \quad \alpha = \sqrt{\frac{m \omega}{\hbar}} = \left( \frac{mk}{\hbar^2} \right)^{1/4} \quad \omega = \sqrt{\frac{k}{m}} \quad N_n = \left( \frac{\alpha}{\sqrt{\pi^2 n!}} \right)^{1/2} \]

(5)

\[ \int_{-\infty}^{\infty} \left( \frac{m \omega}{\hbar} \right)^{1/4} e^{-\frac{m \omega}{2\pi^2} (x-a)^2} u_n(x) dx = \left( \frac{m \omega}{2 \hbar} \right)^{n/2} \frac{a^n}{\sqrt{n!}} e^{-\frac{m \omega a^2}{\hbar}} \]

(6)

\[ e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} \]

(7)

\[ \int_{0}^{\infty} x^n e^{-sx} dx = \frac{n!}{s^{n+1}} \]

(8)

*****************************************************************************

\[ \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} ; \quad \beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} ; \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]

(9)

\[ f(\vec{q}) = -\frac{m}{2\pi \hbar^2} \int V(\vec{r}) e^{i\vec{q} \cdot \vec{r}} d^3r \]

\[ f(\theta) = \frac{1}{2i\hbar} \sum_{l=0}^{\infty} (2l+1)(e^{2i\delta_l} - 1) P_l(\cos \theta) \]

(10)

\[ \frac{d\sigma}{d\Omega} = |f|^2 \quad \sigma = \frac{4\pi}{k} \text{Im}(f(0)) \]

(11)
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PART A

1. Consider the following 3-dimensional wave function in the rectangular coordinate system

\[ \psi(x, y, z) = A (ix + 3z) e^{-\alpha r^2/2} \]

where \( \alpha \) is a constant greater than zero, and where as usual \( r^2 = x^2 + y^2 + z^2 \)

(a) [4] Normalize this wave function by finding \( A \) and calculate \( \langle x \rangle \)

(b) [4] Calculate \( L_z \psi \) where \( L_z \) is the angular momentum operator in the \( z \) direction. Use the rectangular coordinate system.

(c) [7] Write \( x \) and \( z \) in terms of the spherical harmonics and \( r \). After you do that express the wave functions in terms of \( r \) and spherical harmonics.

(d) [5] What are the possible values one can measure for the angular momentum in the \( z \) direction and for the square of the angular momentum. Give the probabilities for obtaining those values.

(e) [5] Calculate \( L_z \psi \) using the wave function obtained in part c

2. Suppose that at time \( t = 0 \) the momentum wave function is given by \( \varphi(p, 0) \) and suppose that for some potential the solution to Schrödinger’s equation in momentum space is given by

\[ \varphi(p, t) = e^{-\frac{i}{\hbar} S(p, t)} \varphi(p - ct, 0) \]

where

\[ S(p, t) = \frac{p^2}{2m} t - \frac{pc}{2m} t^2 + \frac{c^2}{6m} t^3 \]

(a) [9] Staying in the momentum representation calculate \( \langle p \rangle_t \) and \( \langle x \rangle_t \) expressing your answer in terms of \( \langle p \rangle_0 \) and \( \langle x \rangle_0 \).

(b) [8] Suppose the initial wave function in position space is

\[ \psi(x, 0) = \delta(x - x_0) \]

Calculate \( \varphi(p, 0) \) and then calculate \( \psi(x, t) \).

(c) [8] The Schrödinger’s time dependent equation in the momentum representation is

\[ i\hbar \frac{\partial}{\partial t} \varphi(p, t) = \frac{p^2}{2m} \varphi(p, t) + V(x) \varphi(p, t) \]

where \( x \) is the position operator in the momentum representation. Substitute \( \varphi(p, t) \) into it and find \( V(x) \).

Hint: calculate \( i\hbar \frac{\partial}{\partial t} \varphi(p, t) - \frac{p^2}{2m} \varphi(p, t) \) and see what \( V(x) \) must be. In your work you may use the following identity which can be easily verified but you don’t have to do so.

\[ \left( \frac{pc}{m} t - \frac{c^2}{2m} \right) \varphi(p, t) - \frac{h}{i} e^{-\frac{i}{\hbar} S(p, t)} \frac{\partial}{\partial p} \varphi(p - ct, 0) = \frac{h}{i} \frac{\partial}{\partial p} \varphi(p, t) \]
PART B

Problem 3. A two-dimensional harmonic oscillator is governed by the Hamiltonian
\[ H_0 = \hbar \omega \left( a^+ a + b^+ b + 1 \right) \]
where \( a^+ \) and \( a \) are raising and lowering operators for states excited along the \( x \)-direction and \( b^+ \) and \( b \) are raising and lowering operators for states excited along the \( y \)-direction. The oscillator is initially in the ground state. It is subjected to a time-dependent perturbation
\[ H_1(t) = A (a + a^+ + b + b^+) \cos(\omega t) \exp(-\alpha t) \]
where \( A, \omega \), and \( \alpha \) are positive real constants. The full Hamiltonian is
\[ H(t) = H_0 + H_1(t) \].

a) (8 points) Find, to first order in perturbation theory, the wave function for the system at long times \( t \gg 1/\alpha \).

b) (7 points) Find the ground state energy of the full system at time \( t = 0 \). Calculate the first and second-order perturbation theory corrections.

c) (5 points) Find the ground state wave function of the full system at time \( t = 0 \). Calculate only the first-order perturbation correction.

d) (5 points) Find the exact ground state energy of the full system at time \( t = 0 \).

Problem 4. A particle of mass \( m \) moves along the \( z \)-direction with energy \( E \). It encounters a potential field produced by two point scatterers
\[ V(\vec{r}) = -V_0 \left[ \delta(\vec{r}) + \delta(\vec{r} - a\hat{k}) \right] \]
a) (8 points) Use the first Born approximation to obtain the scattering amplitude for scattering through an angle \( \theta \) relative to the initial direction of propagation.

b) (5 points) Find the differential scattering cross section.

c) (3 points) Find the total scattering cross section.

d) (4 points) Under what condition will there exist an angle for which there will be no scattering?

e) (5 points) Find the S-wave phase shift in the limit of low-energy scattering.
\[ \int_{-\infty}^{\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^{\infty} x^2 e^{-ax^2+bx} = \frac{2a+b^2}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^{\infty} x^2 e^{-ax^2} = \frac{\sqrt{\pi}}{2a^{3/2}} \]

\[ x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta, \]

\[ \psi(x, t) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} \varphi(p, t) e^{i px/\hbar} dp \quad ; \quad \varphi(p, t) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} \psi(x, t) e^{-i px/\hbar} dp \]

\[ \int_{-\infty}^{\infty} e^{i(y'-y)x} dx = 2\pi \delta(y - y') \]

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \]

\[ L_x = \frac{\hbar}{i} \left( \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) = \frac{\hbar}{i} \left( \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \varphi} \right) \]

\[ L_y = \frac{\hbar}{i} \left( \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) = \frac{\hbar}{i} \left( \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \varphi} \right) \]

\[ L_z = \frac{\hbar}{i} \left( \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = \frac{\hbar}{i} \frac{\partial}{\partial \varphi} \]

\[ Y_0^0(\theta, \varphi) = \sqrt{\frac{1}{4\pi}} \]

\[ Y_1^0(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos \theta \]

\[ Y_1^{\pm 1}(\theta, \varphi) = \mp \sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin \theta = \pm \sqrt{\frac{3}{8\pi}} \sin \theta (\cos \varphi \pm i \sin \varphi) \]

\[ f(\theta) = -\frac{m}{2\pi \hbar^2} \int e^{-i\mathbf{q} \cdot \mathbf{r}} V(\mathbf{r}) d^3 r, \quad q = \vec{k} - \vec{k} \]

\[ f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) \left[ e^{2i\theta} - 1 \right] P_l(\cos \theta) \]

\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2 - 1}{2}, \ldots \]

\[ a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^* |n\rangle = \sqrt{n+1}|n+1\rangle, \quad [a, a^*] = 1 \]
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PART A

Question 1. This question consists of two independent parts

1a) [13] Consider the following potential problem.

Find the single equation that can be used to obtain the energy eigenvalues. The equation must contain only the following parameters: \( V_1, V_2, L, m, \hbar \) and the energy \( E \). For the region between 0 and \( L \) take the following form for the wave function.

\[
u(x) = C \sin(kx + D) \quad k = \sqrt{\frac{2m}{\hbar^2} E}
\]

This will make the algebra easier.

1b) Consider spin in the \( x - y \) plane making and angle \( \theta \) with the \( x \) axis.

i) [7] Show that the spin matrix is given by \( \sigma_\theta = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \) and find the eigenvalues and eigenvectors of \( \sigma_\theta \).

ii) [5] Suppose the spin state with respect to the \( x \) axis is \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \); what is the probability of measuring spin up in the \( \theta \) direction?

Question 2. This question consists of two independent parts

2a) The Hamiltonian for a rigid rotor is given by

\[
H = -\frac{\hbar^2}{2I} \frac{d^2}{d\varphi^2}
\]

where \( \varphi \) is the azimuthal angle and \( I \) is the moment of inertia.

i) [6] Find the eigenvalues and eigenfunctions of \( H \). Remember that the eigenfunctions must have the same value when one goes through an angle \( 2\pi \).

ii) [10] Using separation of variables show that the general solution to Schrödinger’s equations [ \( H\psi(\varphi, t) = i\hbar \frac{\partial}{\partial t} \psi(\varphi, t) \) ] is given by

\[
\psi(\varphi, t) = \sum_{n=0}^{\infty} c_n T_n(t) u_n(\varphi)
\]

Find explicitly the functions \( T_n(t) \) and \( u_n(\varphi) \).

iii) [4] Find \( \psi(\varphi, t) \) if \( \psi(\varphi, 0) = \sin 2\varphi \cos \varphi \)

2b) [5] Consider the three dimensional problem in which the Hamiltonian is given by

\[
H = H_0 - cL_z
\]

and where \( H_0 \) is the hydrogen atom Hamiltonian, \( c \) is a constant and \( L_z \) is the angular momentum operator in the \( z \) direction. Find the eigenvalues and eigenfunctions of \( H \). Express your answer in terms of the Hydrogen atom, eigenfunctions, \( u_{nlm} \), and eigenvalues, \( E_n \).
1. Suppose that the Hamiltonian for a relativistic electron of mass $m$ is given by

$$H = c\beta \vec{\Sigma} \cdot \vec{p} + \mu \gamma_5$$

where the four-dimensional matrices are given by

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \vec{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad \text{and} \quad \gamma_5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

and $\sigma$ denote the Pauli spin matrices and $I$ is the two-by-two unit matrix.

a) (5 points) What is the value of the parameter $\mu$?

b) (5 points) The electron moves along the positive z-direction with its spin projection parallel to its momentum. What is the eigenvalue and eigenvector of $H$?

c) (5 points) Evaluate the commutator of $H$ with the operator $\vec{\Sigma} \cdot \vec{p}$. Is this operator a constant of the motion?

d) (10 points) Is the vector $\vec{\Sigma}$ a constant of the motion? Is the vector $\vec{L}$ a constant of the motion?

Construct a linear combination of $\vec{\Sigma}$ and $\vec{L}$ that is a constant of the motion. What is the physical meaning of that constant vector?

2. A three-level system is described by the Hamiltonian

$$H_1 = \begin{bmatrix} A & B & 0 \\ B & A & 0 \\ 0 & 0 & C \end{bmatrix}$$

Assume that $C > B > A > 0$ and that $C$ is much larger than both $A$ and $B$.

a) (8 points) Find the eigenvalues and eigenvectors of the system.

b) (5 points) Suppose that the system is prepared in the ground state of $H_1$ and then the Hamiltonian is suddenly switched to

$$H_2 = \begin{bmatrix} B & A & 0 \\ A & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

Find the probability for the system to be found in the new ground state.

c) (5 points) Suppose instead that the system is prepared in the ground state of $H_1$ and then the Hamiltonian is very slowly switched to $H_2$. Find the probability for the system to be found in the new ground state.

d) (7 points) Finally, suppose that the system is prepared at $t = -\infty$ in the ground state of $H_1$ but the system is described by the Hamiltonian

$$H(t) = H_1 + \lambda H_2 \exp(-\alpha t^2)$$

where $\lambda \ll 1$ and $\alpha > 0$. Find, to first order in perturbation theory, the probability that it will be found in the first excited state at $t = +\infty$. 
THE CITY UNIVERSITY OF NEW YORK  
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics  
January 24, 2012

Instructions:  
a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.  
b) There are two parts to this exam, parts A and B. Do two questions but you must chose one problem from each part. Each problem is worth 25 points.  
c) Start each problem on a new page.  
d) Indicate clearly which two problems you choose to solve.

PART A

Question 1. Consider the Schrodinger’s equation for a central potential 

$$\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi = E\psi$$

Write the solution as 

$$\psi = R(r)Y^m_\ell(\theta, \varphi)$$

where $Y^m_\ell(\theta, \varphi)$ are the standard spherical harmonics.

1a) [4] Find the Equation that $R(r)$ satisfies.

1b) [4] Let 

$$u(r) = rR(r)$$

Find the Equation that $u(r)$ satisfies.

1c) [17] Now consider s states, that is, the case where $\ell = 0$. The equation that you derived in part b should reduce to 

$$-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} u + V u = Eu$$

(We are giving you this equation so that you can do this part even if you got parts a) and b) wrong.) Consider the following potential: 

$$V(r) = \begin{cases}  
\infty & 0 < r < a \\
0 & a < r < b \\
V_0 & b < r 
\end{cases}$$

Obtain the single equation that can be used to calculate the energy eigenvalues. The equation should contain only $m, \hbar, a, b, V_0, E$. Take $0 < E < V_0$.  

Hint: in the region $a < r < b$ write the solutions as functions of $r - a$, and in the region $b < r$ write the solution as functions of $r - b$.  

Question 2.

Consider the following Hamiltonian in one dimension

\[ H = \frac{p^2}{2m} - \alpha x + \beta p \]

where \( \alpha \) and \( \beta \) are constants. The aim is to obtain the time dependent momentum wave function \( \varphi(p, t) \), in terms of \( \varphi(p, 0) \). Write the time dependent Schrödinger equation in momentum space

\[ i\hbar \frac{\partial}{\partial t} \varphi(p, t) = H \varphi(p, t) \]  

(1)

and assume we can write a solution in the form

\[ \varphi(p, t) = F(p - \alpha t)e^{iS(p)} \]  

(2)

where \( F \) and \( S \) are functions to be determined. Remember that the position operator in the momentum representation is \( x = i\hbar \frac{\partial}{\partial p} \).

1a) [7] Substitute Eq. (2) into Eq. (1). You will find that you can determine \( S(p) \). Do so.

1b) [5]. Having determined \( S(p) \) let \( t = 0 \) in Eq. (2) and determine \( F(p) \) in terms of \( \varphi(p, 0) \). Write \( \varphi(p, t) \) explicitly in terms of \( \varphi(p, 0) \).

1c) [13] Using your results find the expressions for \( < x >_t \) and \( < p >_t \) in terms so of \( < x >_0 \) and \( < p >_0 \). Do not guess – you must show your work in detail.

Relevant Equations

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]

\[ L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \]
PART B

Problem 3.

An electron with mass $m$ and charge $-e$ is in a quantum dot which is modeled as a three-dimensional harmonic oscillator potential $V(r) = \frac{m}{2} \omega^2 r^2$.

a) [18 points] If a uniform electric field $\vec{E} = E_0 \hat{k}$ is applied along the z-direction find the expectation value for the induced electric dipole in the ground state. The electric dipole moment operator is

$$\hat{\mu} = -e \hat{r}$$

You may solve both parts of this problem either by using perturbation theory or using some other clever technique.

b) [7 points] Suppose that the electric field is turned off and two charges of size $q$ are placed at the positions $z = -a$ and $z = +a$, as shown. You may assume that $a$ is much larger than the size of the oscillator. Compute the expectation value of the quadrupole moment for this case.

$$Q = -\frac{e}{2} (3z^2 - r^2)$$

Again, take the oscillator to be in its ground state. Hint: The electrostatic potential in the neighborhood of the origin is approximately given by $q = 2kq/a + kq(3z^2 - r^2)/a^3 + ...$

Useful formulas:

$$\hat{H} = \left( a^+ a + \frac{1}{2} \right) \hbar \omega, \quad a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^+|n\rangle = \sqrt{n+1}|n+1\rangle, \quad x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^+)$$
Problem 4.

a) [5 points] The hydrogen atom in the 2S state is metastable. Its lifetime against decay is $1/7$ s. On the other hand the 2P state has a lifetime on the order of a nanosecond. Explain in a few sentences why there is this large disparity in lifetimes. [Hint: think about selection rules].

b) [8 points] A physicist prepares hydrogen atoms in the 2S state and places them in a beam of neutrons, moving with wave vector $k$ along the $z$-direction. The interaction between the neutron and the electron may be modeled as the point contact potential. $V(\vec{r}, \vec{r}_n) = V_0 \delta(\vec{r} - \vec{r}_n)$. Find the matrix element for the inelastic transition $\langle 1S, \vec{k}' | V | 2S, \vec{k} \rangle$.

c) [5 points] What are the energies of the initial and final atomic states?

d) [7 points] Find the differential scattering cross section, in the first Born approximation, for the neutron to be inelastically scattered through an angle $\theta$ as the atom makes the 2S to 1S transition. Comment on why it reasonable that this cross section becomes enormous at low neutron energies?

Relevant wave functions for hydrogen are:

$$\psi_{1S} = \frac{1}{\sqrt{\pi a^2}} \exp(-r/a)$$

$$\psi_{2S} = \frac{1}{\sqrt{8\pi a^2}} \left(1 - \frac{r}{2a}\right) \exp\left(-\frac{r}{2a}\right)$$

where $a$ is the first Bohr radius. A useful integral to know is:

$$\int_0^\infty e^{-hx} x^n dx = \frac{n!}{h^{n+1}}$$

Other useful formulas

$$\Gamma = \frac{2\pi}{h} \sum_f |M|^2 \delta(E_f - E_i), \quad \vec{j} = \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*), \quad d\sigma = \frac{d\Gamma}{d\Omega}$$
THE CITY UNIVERSITY OF NEW YORK  
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics  
June 19, 2012

Instructions:
a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.
b) There are two parts to this exam, parts A and B. Do two questions but you must chose one problem from each part. Each problem is worth 25 points.
c) Start each problem on a new page.
d) Indicate clearly which two problems you choose to solve.

PART A

1. Consider the following time dependent Hamiltonian

\[ H = \frac{p^2}{2m} - Ft x \]

where \( F \) is a constant and \( t \) is time.

(a) [10] Using Heisenberg’s equation of motion write and solve the equations for the operators \( p \) and \( x \). That is, find \( p(t) \) and \( x(t) \) in terms of \( p(0) \) and \( x(0) \).

(b) [10] Suppose the state function at time zero is

\[ \psi(x,0) = \left( \frac{\alpha}{\pi} \right)^{1/4} e^{-\alpha x^2/2 + i\beta(x-\gamma)^2/2} \]

where \( \alpha, \beta, \gamma, k \) are real constants. Normalize the wave function and find \( \langle x \rangle_t, \langle p \rangle_t, \langle p^2 \rangle_t \) as functions of time.

(c) [5] Find \( \langle H \rangle_t \). After you obtain your answer check you it by taking \( F = 0 \). Of course, you should get that

\[ \langle H \rangle_t = \frac{\langle p_0^2 \rangle}{2m} \]

If you don’t it means you made a mistake somewhere.
2. Parts a) and b) are independent of each other.

(a) [15] Consider the following three dimensional wave function.

$$\psi(r, \theta, \varphi) = A(u_{n00} + \lambda u_{n10})$$

where $\lambda$ is a constant and

$$u_{n00} = \frac{1}{4\sqrt{2\pi}} \left( \frac{1}{a_0} \right)^{3/2} \left( 2 - \frac{r}{a_0} \right) e^{-r/2a_0}$$

$$u_{n10} = \frac{1}{4\sqrt{2\pi}} \left( \frac{1}{a_0} \right)^{3/2} \frac{r}{a_0} e^{-r/2a_0} \cos \theta$$

Normalize the wave function and find the dipole moment. The dipole moment is defined as $\langle z \rangle$. In your calculations you should take advantage that $u_{n00}$ and $u_{n10}$ and are orthogonal. Also take advantage of symmetry. In particular explain why

$$\int z |u_{n00}|^2 \, d\mathbf{r} = 0$$

$$\int z |u_{n10}|^2 \, d\mathbf{r} = 0$$

(b) [10] Consider the following wave function

$$\psi(\theta, \varphi) = A(3Y_2^{-1}(\theta, \varphi) + 4iY_2^0(\theta, \varphi) + (1 + i)Y_2^1(\theta, \varphi))$$

Normalize the wave function. What are the possible values one can measure of $L^2$ and $L_z$ and what are the respective probabilities. Also calculate $\langle L^2 \rangle$ and $\langle L_z \rangle$. 

PART B

Problem 1

A particle of mass $m$ is moving along the $z$-axis with wave vector $k$. It encounters a spherically symmetric scattering potential given by

$$V(r) = V_0 \Theta(a-r) + V_1 \delta(r-a)$$

where $\Theta(a-r) = \begin{cases} 1 & \text{if } r < a \\ 0 & \text{if } r > a \end{cases}$ and $\delta(r-a)$ is the Dirac delta function.

a) [15] Find the S-wave phase shift.

b) [5] Assuming that only the S-wave contributes significantly to the scattering find an expression for the scattering amplitude.

c) [5] Find the total cross section for the situation described in part b.

Useful formula:

$$f(\theta) = \frac{1}{2i k} \sum_{l=0}^{\infty} (2l + 1) \left[ e^{i \delta} - 1 \right] P_l(\cos \theta)$$

Problem 2

An electron moves in three dimensions in a harmonic oscillator potential. The hamiltonian is given by

$$H_0 = \frac{p^2}{2m} + \frac{m \omega^2 r^2}{2}$$

a) [9] Derive a formula for the energy eigenvalues.

b) [8] Suppose that the following perturbation is introduced

$$H_1 = m g z$$

Where $z$ is the vertical component of the displacement vector. Find the corrected energy eigenvalue for the ground state through second order in perturbation theory.

c) [8] Find the exact ground state energy for the hamiltonian $H = H_0 + H_1$.

Useful formulas:

$$\Delta E_n^{(1)} = \langle n|H_1|n \rangle, \quad \Delta E_n^{(2)} = \sum_{j \neq n} \frac{|\langle j|H_1|n \rangle|^2}{E_n - E_j}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^+), \quad p_x = -i \sqrt{\frac{m\omega \hbar}{2}} (a - a^+)$$

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^+|n\rangle = \sqrt{n+1}|n+1\rangle$$
\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi = E\psi \quad (1) \]
\[ \psi = R(r)Y_{\ell m}(\theta, \varphi) \quad (2) \]
\[ -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) R = ER \quad (3) \]
\[ u = rR \quad (4) \]
\[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} u + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) u = Eu \]

\[ \frac{dA}{dt} = \frac{1}{i\hbar} [A, H] + \frac{\partial A}{\partial t} \quad (5) \]

\[ \int_0^\infty x^n e^{-sx} dx = \frac{n!}{s^{n+1}} \]
\[ \int_{-\infty}^\infty x^2 e^{-ax^2 + bx} dx = \frac{2a + b^2}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \quad (6) \]
THE CITY UNIVERSITY OF NEW YORK
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics
January 22, 2013

Instructions:
a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.
b) There are two parts to this exam, parts A and B. Do two questions but you must choose one problem from each part. Each problem is worth 25 points.
c) Start each problem on a new page.
d) Indicate clearly which two problems you choose to solve.

PART A

1. Consider the three dimension problem where the potential is given by

\[ V(x, y, z) = \begin{cases} \frac{1}{2} k z^2 & \text{For } 0 < x < L_x \text{ and } 0 < y < L_y \\ \infty & \text{otherwise} \end{cases} \]

(a) [12] Solve the eigenvalue problem for the energy eigenfunctions and eigenvalues, that is solve

\[ \frac{-\hbar^2}{2m} \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] u(x, y, z) + V(x, y, z)u(x, y, z) = Eu(x, y, z) \]

Use separation of variables and explain each step of the derivation. Write explicitly the energy eigenvalues and eigenfunctions. (You may assume that you know the solutions to the particle in a box and to the harmonic oscillator problems; the equations are given in the attached equation sheet).

(b) [6] Write explicitly the ground state and the first excited state and their corresponding energies.

(c) [7] For the ground state calculate \( \langle x + y + z \rangle \) and \( \langle x^2 + y^2 + z^2 \rangle \)

2. Starting Schrodinger’s time dependent equation for a free particle in momentum space derive the momentum wave function at time \( t \), \( \varphi(p, t) \), in terms of the initial momentum wave function \( \varphi(p, 0) \).

(a) [8] Suppose the initial momentum wave function is given by

\[ \varphi(p, 0) = Ae^{-a(p - p_0)^2/2 - bi(p - p_0)^2/2\hbar} \] (1)

Normalize and calculate \( \langle x \rangle_t, \langle p \rangle_t, \langle x^2 \rangle_t, \langle p^2 \rangle_t \) as functions of time. (do not attempt to calculate \( \psi(x, t) \) )

(b) [6] (Independent of parts a) and b) ) Write and solve Heisenberg’s equation of motion for the position and momentum operators for a free particle. That is find the position and momentum operators at time \( t \) in terms of the operators at time zero. Do not just write the answers, you must derive them. (It is important to indicate what quantities are operators: we suggest you use the notation of a caret over the symbol to indicate an operator. For example, in the case of position and moment: \( \hat{x}, \hat{p} \))

(c) [12] Using your results in part c) calculate calculate \( \langle x \rangle_t, \langle p \rangle_t, \langle x^2 \rangle_t, \langle p^2 \rangle_t \). Of course they should agree with your results in part b)
PART B

Problem 3.

Consider a hydrogen atom described by the nonrelativistic Hamiltonian

\[ H_0 = \frac{p^2}{2m} - \frac{e^2}{r} \]

In this problem neglect the intrinsic spin of the electron. The orbital magnetic moment operator is given by

\[ \vec{\mu} = -\frac{e}{2mc} \vec{L} \]

The atom is prepared in the eigenstate given by the function

\[ u_1(r, \theta, \varphi) = N r^2 \sin \theta \cos \theta e^{i\varphi} e^{-\alpha r} \]

a) [10] Find the constants \( N \) and \( \alpha \).
b) [5] Find the energy of the atom.
c) [5] Find the expectation value of the magnetic moment vector.

d) [5] Find the shift of the energy of the atom.

Problem 4.

A spin \( \frac{1}{2} \) particle is described by the Hamiltonian

\[ H = \frac{\hbar \omega_0}{2} \sigma_z \]

At time \( t = 0 \) the particle is in a state with its spin projection directed along the negative \( z \) direction. For times \( t > 0 \) it is subjected to the time-dependent interaction

\[ H_1(t) = a \sigma_x e^{-bt} \]

a) [10] Find the wave function as a function of time to first order in perturbation theory.
b) [10] Find the expectation value of \( \sigma_x \) as a function of \( t \) for \( t > 0 \).
c) [5] What is the probability that the spin has flipped its projection for very long times?
Equations

Particle in a box:

\[ E_n = \frac{\pi^2 \hbar^2}{2mL^2} n^2 \quad ; \quad u_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \]

For Harmonic oscillator:

\[ u_n(x) = N_n e^{-a^2 x^2/2} H_n(ax) \]

\[ E_n = (n + \frac{1}{2}) \hbar \omega \quad ; \quad \alpha = \sqrt{\frac{m\omega}{\hbar}} = \left(\frac{mk}{\hbar^2}\right)^{1/4} \quad ; \quad \omega = \sqrt{\frac{k}{m}} \quad ; \quad N_n = \left(\frac{\alpha}{\sqrt{\pi} 2^n n!}\right)^{1/2} \]

\[ u_0(x) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} e^{-a^2 x^2/2} \]

\[ u_1(x) = \sqrt{2a} \left(\frac{\alpha^2}{\pi}\right)^{1/4} xe^{-a^2 x^2/2} \]

\[ \int_{-\infty}^{\infty} e^{-\alpha^2 x^2 + bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^{\infty} x e^{-\alpha^2 x^2 + bx} = \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2 + bx} = \frac{2a + b}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{0}^{\infty} x^n e^{-sx} \, dx = \frac{n!}{s^{n+1}} \]

\[ L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \]

\[ L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \]

\[ H = -\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \sin \theta \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] - \frac{e^2}{r} \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma} = \frac{\hbar}{2} \hat{\sigma} \]

\[ S_{\pm} = S_x \pm i S_y, \quad S_z | s, m_s \rangle = \hbar m_s | s, m_s \rangle, \quad S_{\pm} | s, m_s \rangle = \hbar \sqrt{(s \mp m_s)(s \pm m_s + 1)} | s, m_s \pm 1 \rangle \]
Instructions:
a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.
b) There are two parts to this exam, parts A and B. Do two questions but you must chose one
from each part. Each question is worth 25 points.
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PART A

1. Parts a,b,c are independent of each other.

   (a) [10] Consider the following time dependent momentum wave function

   \[ \varphi(p, t) = Ne^{-\alpha(p-\beta t)^2/2-iA(p, t)/\hbar} \]

   \[ A(p, t) = \frac{p^2 t}{2m} - \frac{p\beta t^2}{2m} + \frac{\beta^2 t^3}{6m} \]

   where \( \alpha, \beta \) are real numbers. Normalize \( \varphi(p, t) \) and calculate \( \langle x \rangle_t, \langle p \rangle_t, \langle x^2 \rangle_t, \langle p^2 \rangle_t \).
   Also, calculate the uncertainty product \((\Delta x)(\Delta p)\) and show that it is greater then \( \hbar/2 \) for all time.

   (b) [10] Consider the following Hamiltonian

   \[ H = \frac{p^2}{2m} + \alpha p \]

   where \( \alpha \) is a real constant. Obtain the Greens function – that is the \( G(x, x', t) \) for which

   \[ \psi(x, t) = \int G(x, x', t)\psi(x', 0)dx' \]

   Hint: follow the same derivation that is standardly done for the free particle.

   (c) [5] Suppose the function \( \psi(x, t) \) satisfies Schrödinger time dependent equation

   \[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = H\psi(x, t) \tag{1} \]

   Now consider the wave function

   \[ u(x, t) = R(x, t)\psi(x, t) \]

   where \( R \) is an operator that depends on space and time. Derive the operator equation that \( R \) must satisfy so that \( u(x, t) \) is also a solution of Eq. (1). [ This was first done by Lewis and Reisenfeld in 1969]
2. Parts a, b, c are independent of each other.

(a) [9] Consider the potential

\[ V(r) = -\frac{e^2}{r} + \frac{1}{2}\omega^2 r^2 \]

which is the potential for the hydrogen atom plus the isotropic harmonic oscillator. Considering \( \frac{1}{2}\omega r^2 \) as a perturbation on the hydrogen atom use first order perturbation theory to find the \( \omega \) that gives zero total energy for the ground state energy.

(b) [8] Consider the following potential

\[ V(r) = \frac{\hbar^2}{2m} \left( -\frac{5\alpha}{r} + \frac{7}{4} r \right) \]

where \( \alpha \) is a fixed constant and the following wave function

\[ \psi(\theta, \varphi, r) = Y^m_\ell(\theta, \varphi) r^\beta e^{-\alpha r} \]

with \( m = 0 \) and \( \ell = 1 \). Is there a value for \( \beta \) which makes \( u(\theta, \varphi, r) \) an eigenfunction of the Hamiltonian? If there is find it and the corresponding energy eigenvalue.

(c) [8] Suppose \( \psi_n \) are the eigenfunctions of the Hamiltonian \( H \) with corresponding energy eigenvalues

\[ H\psi_n = E_n\psi_n \]

and suppose the wave function \( \psi \) is defined by

\[ \psi = A(\psi_n + \lambda u) \]

where \( \lambda \) is a real small parameter and \( u \) is a function. Show that the expected value of the Hamiltonian taken with \( \psi \) is second order in \( \lambda \). For convenience you may assume that \( \psi_n \) and \( u \) are real and normalized and define \( \alpha = \int \psi_n u dx \). Hint: Be sure to normalize \( \psi \).
Equations

\[ \int_{\infty}^{-\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \quad (2) \]

\[ \int_{\infty}^{-\infty} xe^{-ax^2+bx} = \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \quad (3) \]

\[ \int_{\infty}^{-\infty} x^2 e^{-ax^2+bx} = \frac{2a+b^2}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \quad (4) \]

\[ \int_{\infty}^{-\infty} x^4 e^{-ax^2} = \frac{3}{4a^{3/2}} \quad (5) \]

\[ \int_{0}^{\infty} x^n e^{-ax^2} dx = \frac{n!}{a^{n+1}} = \begin{cases} 
1/a & n = 0 \\
1/a^2 & n = 1 \\
2/a^3 & n = 2 \\
6/a^4 & n = 3 \\
24/a^5 & n = 4 
\end{cases} \quad (6) \]

Hydrogen atom ground state:

\[ \psi_0 = e^{-r/a_0} \sqrt{\frac{\pi}{a_0^3}}, \quad E_0 = -\frac{m e^4}{2\hbar^2}, \quad a_0 = \frac{\hbar^2}{m c^2} \quad (7) \]

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (8) \]

\[ -\frac{\hbar^2}{2m r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \left( V + \frac{\hbar^2}{2mr^2} \ell (\ell + 1) \right) R = ER \quad (9) \]

For central potentials:

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r) \psi = E \psi \quad (10) \]

\[ \psi(r, \theta, \varphi) = R(r) Y^m_l(\theta, \varphi) \quad (11) \]

\[ L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \quad (12) \]

\[ -\frac{\hbar^2}{2m r^2} \frac{d}{dr} r^2 \frac{dR}{dr} + \left( V(r) + \frac{\hbar^2}{2mr^2} \ell (\ell + 1) \right) R = ER \quad (13) \]

If

\[ u = rR \quad (14) \]

then

\[ -\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left( V + \frac{\hbar^2}{2mr^2} \ell (\ell + 1) \right) u = Eu \quad (15) \]
Problem 1

The potential seen by electrons at the surface of a metal placed in a strong electric field has the form shown in the Figure (The potential in the region $x > 0$ is given by

$$V(x) = \begin{cases} 
-Fx, & 0 < x < b \\
-Fb, & x > b 
\end{cases}$$

Consider an electron of energy $\varepsilon < 0$ (see the Figure) incident on the surface from inside of the metal ($x < 0$).

---

a) Write down expressions for the quasi-classical wave function of the electron in regions $x < 0$, $0 < x < x_0$, $x > x_0$, where $x_0$ is the classical turning point (find it in terms of energy and $F$) (5 points)

b) Find connections between coefficients of the wave function in each of these three regions using WKB connection formulas. (7 points)

c) Calculate the transmission coefficient of this electron neglecting exponentially small reflection at point $x = b$ across the barrier in the quasi-classical approximation by applying connection formulas at each of the classical turning points (8 points)

d) Determine the limits of applicability of the WKB approximation in this case. (5 points)
Problem 2

A hydrogen atom in a state with the principal quantum number $n = 2$ is in a magnetic and an electric field directed perpendicularly to one another so that the interaction Hamiltonian is given by

$$H_{\text{int}} = q\mathcal{E}x + \frac{eB}{2mc}(L_z + 2S_z)$$

Assume that the fields are strong (energy of the electron in the external electric and magnetic fields is much larger than the energy of the spin-orbit interaction).

a). Make a list of all degenerate states belonging to this energy level. Are they orthogonal to each other? (3 points)

b). Indicate which of the degenerate states are coupled by the perturbation and write down the perturbation matrix in the basis of these states. (8 points)

c). Find the energies of all energy levels with $n = 2$ (coupled and uncoupled) in the first order of the perturbation theory. (8 points)

d) If the electric and magnetic fields are weak compared to the spin-orbit interaction, list the eigenstates of the zero order Hamiltonian corresponding to $n = 2$, which you would use as an initial basis for the perturbation theory. Indicate which states are degenerate. (6 points)
Useful formulas

Normalized wave function of the hydrogen atom with $n = 2$ are:

\[
\psi_{200} = \frac{1}{4a_0^{3/2}\sqrt{2\pi}} \left(2 - \frac{r}{a_0}\right)e^{-r/2a_0}
\]

\[
\psi_{210} = \frac{1}{4a_0^{3/2}\sqrt{2\pi}} \frac{r}{a_0} e^{-r/2a_0} \cos \theta
\]

\[
\psi_{2\pm1} = \pm \frac{1}{8a_0^{3/2}\sqrt{\pi}} \frac{r}{a_0} e^{-r/2a_0} \sin \theta e^{\pm \phi}
\]

Useful integral:

\[
\int_0^\infty dx x^n e^{-x} = n!
\]

Connection formulas for WKB approximation:

\[
\frac{1}{\sqrt{\kappa(x)}} \exp \left[ -\frac{1}{\hbar} \int_a^x \kappa(x') dx' \right] \Rightarrow \frac{2}{\sqrt{p(x)}} \cos \left[ \frac{1}{\hbar} \int_x^a p(x') dx' - \frac{\pi}{4} \right]
\]

\[
\frac{1}{\sqrt{\kappa(x)}} \cos \left[ \frac{1}{\hbar} \int_x^a p(x') dx' + \frac{\pi}{4} \right] \Rightarrow \frac{1}{\sqrt{\kappa(x)}} \exp \left[ \frac{1}{\hbar} \int_x^a \kappa(x') dx' \right]
\]

\[
\frac{1}{\sqrt{\kappa(x)}} \exp \left[ -\frac{1}{\hbar} \int_x^b \kappa(x') dx' \right] \Rightarrow \frac{2}{\sqrt{p(x)}} \cos \left[ \frac{1}{\hbar} \int_b^x p(x') dx' - \frac{\pi}{4} \right]
\]

\[
\frac{1}{\sqrt{p(x)}} \cos \left[ \frac{1}{\hbar} \int_b^x p(x') dx' + \frac{\pi}{4} \right] \Rightarrow \frac{1}{\sqrt{\kappa(x)}} \exp \left[ \frac{1}{\hbar} \int_x^b \kappa(x') dx' \right]
\]
Instructions:
a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.
b) There are two parts to this exam, parts A and B. Do two questions but you must chose one from each part. Each question is worth 25 points.
c) Start each problem on a new page.
d) Indicate clearly which two problems you choose to solve.
PART A

1. Parts a,b,c are independent of each other.

   (a) [7] Starting with Schrödinger time dependent equation derive the equation of motion for the expectation value of an operator, $A$:

   $$\frac{d\langle A \rangle}{dt} = \frac{1}{i\hbar} \langle [A, H] \rangle + \left\langle \frac{\partial A}{\partial t} \right\rangle$$

   Then, show that for Hamiltonians of the form $H = \frac{p^2}{2m} + V(x)$, that the expectation values for position and momentum are

   $$\frac{d\langle x \rangle}{dt} = \frac{1}{m} \langle p \rangle$$

   $$\frac{d\langle p \rangle}{dt} = -\left\langle \frac{\partial V}{\partial x} \right\rangle$$

   Explain in detail your simplification of the commutators.

   (b) [5] Suppose we are solving a bound state problem and at a certain point a the potential behaves as

   $$V(x) = -\lambda \delta(x - a)$$

   Show that the discontinuity of the eigenfunctions at $x = a$ satisfies

   $$\frac{du}{dx}\bigg|_{a^+} - \frac{du}{dx}\bigg|_{a^-} = -\lambda \frac{2m}{\hbar^2} u(a)$$

   (c) [13] Consider the standard problem of a particle in box (0 to L) but with an additional attractive delta function potential $V(x) = -\lambda \delta(x - L/2)$ with $\lambda$ positive. Derive the algebraic equation that can be used to find the energy eigenvalues. Your equation should contain only $m, \hbar, L, E, \lambda$. Hint: Write Schrödinger’s equation and for the region $0 \leq x \leq L/2$ take $u(x) = A \sin kx + B \cos kx$ and for the region $L/2 \leq x \leq L$ take $u(x) = C \sin k(x - L) + D \cos k(x - L)$

2. Parts a and b are independent of each other.

   (a) Consider the following wave function

   $$\psi(r, 0) = 3u_{100}(r) + 4iu_{211}(r)$$

   where $u_{100}(r)$ and $u_{211}$ are the eigenfunctions of the hydrogen atom for the $n, l, m$ states. They are given in the equation sheet.

   i. [8] Normalize the wave and calculate $\langle H \rangle$ and $\langle L^2 \rangle$. You do not have to do any integrals for this part, just use the fundamental properties of $H$ and $L^2$.

   ii. [6] Calculate $\langle r \rangle$.

   (b) [11] Consider the three dimensional anisotropic harmonic oscillator where the potential

   $$V(x, y, z) = \frac{1}{2} m \left( \omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2 \right)$$

   Write the ground state wave function, the ground state energy and calculate $\langle r^2 \rangle$ for the ground state.
Equations for Part A

\[
\begin{align*}
  u_{100} &= \frac{1}{\sqrt{\pi}} \left( \frac{1}{a_0} \right)^{3/2} e^{-r/a_0} \\
  u_{211} &= \frac{1}{8} \frac{I}{\pi} \left( \frac{1}{a_0} \right)^{3/2} \frac{r}{a_0} e^{-r/2a_0} \sin \theta e^{i\phi}
\end{align*}
\]

Hydrogen atom energy:

\[
E_n = -\frac{e^2}{2a_0 n^2} = -\frac{me^4}{2\hbar^2 n^2}
\]

***********************

Harmonic Oscillator:

\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right) u_n(x) = E_n u_n(x)
\]

\[
\begin{align*}
  u_n(x) &= N_n e^{-\alpha^2 x^2 / 2} H_n(\alpha x) \\
  u_n(x) &= \left( \frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} e^{-\alpha^2 x^2 / 2} H_n(\alpha x) \\
  E_n &= (n + \frac{1}{2}) \hbar \omega \\
  \alpha &= \sqrt{\frac{m\omega}{\hbar}} = \left( \frac{mk}{\hbar^2} \right)^{1/4} \\
  \omega &= \sqrt{\frac{k}{m}} \\
  N_n &= \left( \frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} \\
  u_0(x) &= \left( \frac{m\omega}{\hbar\pi} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}
\end{align*}
\]

***********************

Integrals:

\[
\begin{align*}
  \int_0^\pi \sin x \, dx &= 2 \\
  \int_0^\pi \sin^2 x \, dx &= \pi / 2 \\
  \int_0^\pi \sin^3 x \, dx &= 4 / 3 \\
  \int_0^\infty x^n e^{-ax} \, dx &= \frac{n!}{a^{n+1}} \\
  \int_{-\infty}^\infty e^{-ax^2 + bx} \, dx &= \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \\
  \int_{-\infty}^\infty x e^{-ax^2 + bx} \, dx &= \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \\
  \int_{-\infty}^\infty x^2 e^{-ax^2 + bx} \, dx &= \frac{2a + b^2}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}
\end{align*}
\]
Part B

1. A particle of spin $\frac{1}{2}$ and magnetic moment $\mu$ moves in the non-homogeneous magnetic field

$$B_x = B_0 + kz, B_y = -ky, B_z = 0$$

($x, y, z$ are Cartesian components of the position vector)

a) Write down Hamiltonian of the system as a $2 \times 2$ matrix in the basis of the eigenstates of the $z$-component of the spin including only spin part of interaction with the magnetic field (3 pts)

b) Show that upon transformation to the Heisenberg picture, the Hamiltonian does not change (3 pts)

c) Write down Heisenberg equations for Cartesian $x, y, z$ components of the position operator, momentum operator, and spin operators. (7 pts)

d) Solve Heisenberg equations for the spin operators neglecting the inhomogeneous part of the magnetic field (6 pts)

e) Substituting the found results for the spin matrices into equations for the momentum operators, find their solutions. This will give you solution of the problem in the linear with respect to parameter $k$ approximation (6 pts)

2. Consider states of the hydrogen atom with principal quantum number $n = 2$. (Neglect spin)

a) Using degenerate perturbation theory find corrections to the energies of all these states due to static uniform electric field $F$ directed in $z$-direction. (7 points)

b) Find the correct forms of the zero order wave functions for each of these states (7 points)

c) Now imagine that you want to observe the energy splitting induced by the electric field (Stark effect) by measuring optical absorption using monochromatic linearly polarized light. Assuming that the atom is in its ground state and the light is polarized in $z$-direction, determine frequencies of spectral lines in the absorption spectrum, which will be observed, and their relative strength (7 points)

d) Can Stark effect be observed with $x$-polarized light? Give arguments based on Wigner-Eckart theorem (4 points)

Useful formulas for Part B

Normalized wave function of the hydrogen atom with $n = 2$ are:
\[ \psi_{200} = \frac{1}{4a_0^{3/2}\sqrt{2\pi}} \left( 2 - \frac{r}{a_0} \right) e^{-r/2a_0} \]

\[ \psi_{210} = \frac{1}{4a_0^{3/2}\sqrt{2\pi}} \frac{r}{a_0} e^{-r/2a_0} \cos \theta \]

\[ \psi_{21\pm1} = \pm \frac{1}{8a_0^{3/2}\sqrt{\pi}} \frac{r}{a_0} e^{-r/2a_0} \sin \theta e^{\pm ip} \]

Useful integral:

\[ \int_0^\infty dx x^n e^{-x} = n! \]
THE CITY UNIVERSITY OF NEW YORK  
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics  
June 16, 2014

Instructions:

a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.
b) There are two parts to this exam, parts A and B. Do two questions but you must chose one from each part. Each question is worth 25 points.
c) Start each problem on a new page.
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PART A

1. Parts a and b are independent of each other.

   (a) [13] The continuity theorem in quantum mechanics states $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$ where $\rho$ is the probability density and $\mathbf{j}$ is the quantum mechanical current. There is another continuity equation that applies to energy density

   $$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \mathbf{S} = 0$$

   where the energy density is defined as

   $$\epsilon(r, t) = \frac{\hbar^2}{2m} \nabla \psi(r, t) \nabla \psi^*(r, t) + V(r) \psi(r, t) \psi^*(r, t)$$

   and $\mathbf{S}$ is the energy density current. Find $\mathbf{S}$ – the proof goes along the same lines as for the continuity equation. After you obtain your result be sure to show that it is real. You may assume that $V$ is multiplicative. To make it easier do it in one dimension

   $$\epsilon(x, t) = \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x} \frac{\partial \psi^*}{\partial x} + V(x) \psi(x, t) \psi^*(x, t)$$

   $$\frac{\partial \epsilon}{\partial t} + \frac{\partial S}{\partial x} = 0$$

   Also, the following relation may be useful

   $$\left( \frac{\partial}{\partial x} \frac{\partial \psi^*}{\partial t} \right) \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial x} \right) - \frac{\partial \psi^*}{\partial t} \frac{\partial^2 \psi}{\partial x^2}$$

   (b) [12] Consider the following wave function

   $$\psi(x) = \left( \frac{\alpha^2}{\pi} \right)^{1/4} e^{-\alpha^2 x^2/2 + i\beta x + i\gamma x}$$

   under a harmonic oscillator force, where $\beta$ and $\gamma$ are real and as usual $\alpha = \sqrt{\frac{\pi \hbar}{m}}$. What is the probability that a measurement will find the oscillator in the ground state and what is the probability to find it in an excited state. Make sure you express your final answers in forms that are manifestly real.
2. Parts a and b are independent of each other.

(a) Consider the following operator

\[ A = \alpha(L_x^2 + L_y^2) + \beta L_z^2 \]

with \( \alpha \) and \( \beta \) real.

i. [6] Are the spherical Harmonics \( Y_m^{\ell}(\theta, \varphi) \) eigenfunctions of the operator? If yes find the eigenvalues. Hint: first express the operator in terms of \( L^2 \) and \( L_z \).

ii. [6] Consider the wave function

\[ \psi(\theta, \varphi) = \sin \theta \cos \varphi \]

Express this wave function in terms of spherical harmonics and determine if it is an eigenfunction of \( A \). If it is find the eigenvalue. Hint write

\[ \psi(\theta, \varphi) = \sin \theta \cos \varphi = \sin \theta \frac{e^{i\varphi} + e^{-i\varphi}}{2} \]

and use the fact that

\[ Y_{1}^{\pm 1}(\theta, \varphi) = \pm \sqrt{\frac{3}{8\pi}} e^{i\varphi} \sin \theta \]

(b) [13] Using the variational principle estimate the ground state wave function and energy of the hydrogen atom using the following finite extent trial wave function

\[ \psi(r) = \begin{cases} 1 - r/R & \text{if } r \leq R, \\ 0 & \text{if } r > R. \end{cases} \]

where \( R \) is the variational parameter.
Equations for Part A

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} . \]

Hydrogen atom energy:

\[ E_n = - \frac{e^2}{2a_0 n^2} = - \frac{me^4}{2\hbar^2 n^2} \]

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Harmonic Oscillator:

\[ \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2 \right) u_n(x) = E_n u_n(x) \]

\[ u_n(x) = N_n e^{\alpha x^2/2} H_n(\alpha x) \]

\[ u_n(x) = \left( \frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} e^{\alpha x^2/2} H_n(\alpha x) \]

\[ E_n = (n + 1) \hbar \omega \quad \alpha = \sqrt{\frac{m \omega}{\hbar}} = \left( \frac{mk}{\hbar^2} \right)^{1/4} \quad \omega = \sqrt{\frac{k}{m}} \quad N_n = \left( \frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} \]

\[ u_0(x) = \left( \frac{\alpha^2}{\pi} \right)^{1/4} e^{-\alpha x^2/2} \]

Integrals:

\[ \int_0^\pi \sin x \, dx = 2 \]

\[ \int_0^\pi \sin^2 x \, dx = \pi/2 \]

\[ \int_0^\pi \sin^3 x \, dx = 4/3 \]

\[ \int_0^\infty x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}} \]

\[ \int_{-\infty}^\infty e^{-ax^2 + bx} \, dx = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^\infty xe^{-ax^2 + bx} \, dx = \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^\infty x^2 e^{-ax^2 + bx} \, dx = \frac{2a + b^2}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]
Y_0^0(\theta, \varphi) = \sqrt{\frac{1}{4\pi}}

Y_1^0(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos \theta

Y_1^{\pm 1}(\theta, \varphi) = \pm \sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin \theta

Y_2^0(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)

Y_2^{\pm 1}(\theta, \varphi) = \pm \sqrt{\frac{15}{8\pi}} e^{\pm i\varphi} \sin \theta \cos \theta

Y_2^{\pm 2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\varphi} \sin^2 \theta
1. A particle of spin ½ and magnetic moment moves in the uniform magnetic field

\[ \mathbf{B}(t) = \begin{cases} B_0 \mathbf{k}, & 0 \leq t \leq T \\ 0, & t < 0, t > T \end{cases} \]

(\( \mathbf{k} \) is a unit vector in the direction of the field)

a) Obtain Heisenberg equations for the components of the spin operator of the particle (5 points)

b) By solving the Heisenberg equations, find operators \( \hat{S}_{x,y,z}(t) \) in the Heisenberg picture (10 points)

c) If the system is prepared in state described by the spinor

\[ |\chi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

written in the basis of the eigenspinors of the operator \( S_z \) defined at \( t < 0 \), find the probabilities of obtaining various outcomes, if x-component of the spin is measured at \( t > T \) (10 points)

2. Consider a quantum system described by the Hamiltonian of the following form

\[ H = \frac{\hat{p}^2}{2M} + \frac{1}{2} M \omega^2 x^2 + \frac{1}{3!} \kappa x^3 \quad (1) \]

where \( \hat{p} \) and \( \hat{x} \) are canonic operators of momentum and coordinate respectively, \( M \) is mass of the particle, and \( \omega \) is the frequency of harmonic oscillations.

a) Present the Hamiltonian in terms of ladder operators \( \hat{a}, \hat{a}^\dagger \) of the harmonic oscillator

b) Rewrite the Hamiltonian found in a) in the normally ordered form

c) Find the expectation value of energy of the particle described by this Hamiltonian if the system is in a coherent state \( |\alpha\rangle \).

Useful formulas for Part B

\[ a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + i \frac{p}{m\omega} \right); \]

\[ H = -\mathbf{\mu} \cdot \mathbf{B}; \quad \mathbf{\mu} = \frac{e}{mc} \mathbf{S}; \quad \frac{d\hat{A}}{dt} = \frac{i}{\hbar} \left[ \hat{A}, \hat{H} \right] \]
THE CITY UNIVERSITY OF NEW YORK
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics
January 20, 2015

Instructions:
a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.
b) There are two parts to this exam, parts A and B. Do two questions but you must choose one from each part. Each question is worth 25 points.
c) Start each problem on a new page.
d) Indicate clearly which two problems you choose to solve.

PART A

1. Parts a, b, and c are independent of each other.
   
   (a) Consider the following 3 dimensional wave function
   \[ \psi(r, \theta, \varphi) = u_1(r, \theta, \varphi) + \eta u_2(r, \theta, \varphi) \]
   where
   \[ u_1(r, \theta, \varphi) = \frac{1}{4\sqrt{2\pi}} \left( \frac{1}{a_0} \right)^{3/2} \left( 2 - \frac{r}{a_0} \right) e^{-r/(2a_0)} \]
   \[ u_2(r, \theta, \varphi) = \frac{1}{4\sqrt{2\pi}} \left( \frac{1}{a_0} \right)^{3/2} \frac{r}{a_0} e^{-r/(2a_0)} \cos \theta \]
   and where \( \eta \) is a real number. \( u_1(r, \theta, \varphi) \) and \( u_2(r, \theta, \varphi) \) are normalized hydrogen atom eigenfunctions. Normalize the wave function, \( \psi(r, \theta, \varphi) \), and calculate the dipole moment \( \mu = \langle z \rangle \).
   Some of the integrals may be “obviously” zero – if that is the case be sure you explain why they are zero.
   
   (b) Consider the one dimensional problem where the potential is given by
   \[ V(x) = \begin{cases} 
   -\lambda & x > 0 \\
   \infty & x \leq 0
   \end{cases} \]
   and where \( \lambda \) is a given positive constant. For the following unnormalized wave function
   \[ u(x) = x^n e^{-\alpha x} \]
   are there values of \( \alpha \) and \( n \) which makes \( u(x) \) an energy eigenfunction? If yes find \( \alpha \) and \( n \) and find the energy eigenvalue.
   
   (c) Consider the following Hamiltonian
   \[ H = \frac{p^2}{2m} - Fx + Gp \]
   where \( F \) and \( G \) are real numbers. Solve for the eigenfunctions and eigenvalues of \( H \): Do it in the momentum representation. That is solve \( H u_E(p) = E u_E(p) \) for \( u_E(p) \). Remember that in the momentum representation \( x = -\hbar \frac{d}{dp} \). Be sure you appropriately normalize the eigenfunctions. Are the eigenvalues continuous or discreet?
2. Parts a, b and c are independent of each other.

(a) Consider the three dimensional potential given by

\[ V(r) = kr \]

Use the variational principle to estimate the ground state energy. Use the trail wave function

\[ \psi(r, \theta, \varphi) = e^{-\alpha r} \]

where \( \alpha \) is the variational parameter.

(b) Suppose we have a Hamiltonian of the form \( H = \frac{p^2}{2m} + V(x) \) with corresponding eigenvalues and eigenfunctions \( E_n \) and \( u_n(x) \). Define

\[ p_{nk} = \int u_n^*(x)p u_k(x)dx \quad ; \quad x_{nk} = \int u_n^*(x)x u_k(x)dx \]

Find an expression that relates \( p_{nk} \) to \( E_n, E_n \) and \( x_{nk} \). HINT: first prove that \( p = \frac{im}{\hbar} [H, x] \)

(c) Suppose the spin wave function with respect to the \( z \) axis is given by

\[ \psi = \left( \begin{array}{c} a + ib \\ c \end{array} \right) \]

where \( a, b, c \) are real numbers. Normalize the wave function and determine the probability of measuring spin up and down. Calculate the average value of spin in the \( z \) direction and the standard deviation.
Relevant Equations

\[ \frac{d\mathbf{A}}{dt} = \frac{i}{\hbar} [\mathbf{H}, \mathbf{A}] \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

\[ \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} = \begin{cases} \frac{1}{a} & n = 0 \\ \frac{1}{a^2} & n = 1 \\ \frac{2}{a^3} & n = 2 \\ \frac{6}{a^4} & n = 3 \\ \frac{24}{a^5} & n = 4 \end{cases} \]

\[ \int_{-\infty}^\infty e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^\infty xe^{-ax^2+bx} = \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^\infty x^2 e^{-ax^2+bx} = \frac{2a + b^2}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_0^\pi \sin x dx = 2 \]

\[ \int_0^\pi \sin^2 x dx = \frac{\pi}{2} \]

\[ \int_0^\pi \sin^3 x dx = \frac{4}{3} \]

\[ \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \]

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \]

Hydrogen atom energy:

\[ E_n = -\frac{e^2}{2a_0 n^2} = -\frac{me^4}{2\hbar^2 n^2} \]
QM, Part B

Problem 1

a) Use the WKB approximation to determine the bound-state energies of the potential well

\[
V(x) = \begin{cases} 
V_0 \frac{|x|}{a}, & |x| \leq a \\
V_0 = \frac{\hbar^2}{ma^2}, & |x| > a 
\end{cases}
\]

b) An electron is inside of a quantum dot modeled as an infinite spherical potential well:

\[
V(r) = \begin{cases} 
0, & r < R \\
\infty, & r > R 
\end{cases}
\]

Using degenerate perturbation theory, find the first order corrections to the energy of states characterized by orbital number \( l = 2 \) due to a uniform electric field \( \mathbf{F} \) and corresponding zero-order wave functions.

Problem 2

Consider an exciton in a semiconductor as a particle with spin \( s = 1 \) and assume that its orbital state is characterized by orbital momentum \( l = 1 \).

a) What are the possible values of the total angular momentum of the exciton \( \mathbf{J} = \mathbf{L} + \mathbf{S} \) (number \( j \) characterizing the eigenvalues of the operator \( J^2 \))?

b) A measurement of the total angular momentum produced values \( j = 2; m_j = -2 \), where \( m_j \) is the quantum number characterizing eigenvalues of the operator \( J_z \). What are the probabilities of obtaining various values of \( m_L \) and \( m_S \), characterizing eigenvalues of operators \( L_z \) and \( S_z \) if these quantities are measured immediately after the first measurement?

c) Answer the same question as above but assuming that the results of the first measurement produced \( j = 2; m_j = 0 \)

Some useful formulas
\[ Y_2^0 (\theta, \varphi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} (3 \cos^2 \theta - 1); \quad Y_2^{1\pm} (\theta, \varphi) = \pm \frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{\pm i\varphi} \]

\[ Y_2^{1\pm} (\theta, \varphi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{\pm 2i\varphi} \]

Recursion relation for Clebsch-Gordan coefficients

\[
\sqrt{(j \mp m_j)(j \mp m_j + 1)} \langle l, s, m_l, m_s | l, s, j, m_j \mp 1 \rangle = \sqrt{(l \mp m_l)(l \mp m_l + 1)} \langle l, s, m_l, m_s + 1 | l, s, j, m_j \rangle + \\
\sqrt{(s \mp m_s)(s \mp m_s + 1)} \langle l, s, m_l, m_s | l, s, j, m_j \rangle
\]

\[ \langle l, s, l, s | l, s, l + s, l + s \rangle = 1 \]
Instructions:
a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.
b) There are two parts to this exam, parts A and B. Do two questions but you must chose one from each part. Each question is worth 25 points.
c) Start each problem on a new page.
d) Indicate clearly which two problems you choose to solve.

PART A

1. Parts a, b, and c are independent of each other.

(a) [10] A particle is constrained to move on the surface of an infinite cylinder of radius $R$. The central axis of the cylinder is along the z axis. Find the stationary state energy eigenfunctions and eigenvalues. Hint: the Laplacian in cylindrical coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. $$

but since it is constrained to $r = R$ we can take it to be

$$\nabla^2 = \frac{1}{R^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. $$

Hence you have to solve Schrödinger’s equation

$$-\frac{\hbar^2}{2m} \nabla^2 u(\varphi, z) = Eu(\varphi, z)$$

for the eigenfunctions and eigenvalues

(b) [10] Consider the following angular momentum wave function

$$\psi(\theta, \varphi) = N \left[ \alpha Y_m^l(\theta, \varphi) + i\beta Y_l^l(\theta, \varphi) \right]$$

with $\alpha$ and $\beta$ real. Normalize the wave function and calculate the $(\Delta L_z)^2 = \langle L_z^2 \rangle - \langle L_z \rangle^2$. After you obtain your answer show that it is manifestly positive, that is $(\Delta L_z)^2 = (\text{something real})^2$

(c) [5] Suppose the solution to the time dependent Schrödinger equation is given by

$$\psi(x, t) = \int K_x(x, x', t)\psi(x', 0)dx'$$
where $\psi(x,t)$ is the wave function at time $t$ and $\psi(x,0)$ is the wave function at time zero. $K_x$ is called the propagator. Suppose, also, that the momentum wave function, $\varphi(p,t)$, can be written as

$$\varphi(p,t) = \int K_p(p,p',t)\varphi(p',0)dp'$$

where $K_p(p,p',t)$ is the propagator in momentum space. Find the relation between $K_p(p,p',t)$ and $K_x(x,x',t)$.

2. Parts a, b, and c are independent of each other

(a) [10] Consider the following wave function

$$u(r,\theta,\varphi) = R(r)Y_l^m(\theta,\varphi)$$

where

$$R(r) = \frac{\sin \alpha r - \alpha r \cos \alpha r}{\alpha^2 r^2}$$

Are there values for $\alpha, l, m$ for which $u(r,\theta,\varphi)$ is an eigenfunction of some central potential? If yes find the energy eigenvalue, the potential, and determine $l$. [Note: to save you some algebra you may use the following: $\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R(r) = \left( \frac{\alpha^2}{r^2} - \alpha^2 \right) R(r)$]

(b) [10] Consider the following central potential

$$V(r) = -\frac{\lambda}{r^{3/2}}$$

where $\lambda$ is positive. Use the variational principle to estimate the ground state energy. For the trial function take

$$\psi(r,\theta,\varphi) = e^{-\alpha r/2}$$

where $\alpha$ is the variational parameter. [Note: to save you some algebra you may use the following: $\int_0^\infty e^{-\alpha r/2} \left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} e^{-\alpha r/2} \right) 4\pi r^2 dr = \frac{2\pi}{\alpha}$]

(c) [5] Consider the time dependent Schrödinger equation for a free particle in the momentum representation. Call the time dependent solution $\varphi(p,t)$. Given $\varphi(p,0)$ find $\varphi(p,t)$. Then, staying in the momentum representation, show that expected value of position at time $t$, $\langle x \rangle_t$, is given by

$$\langle x \rangle_t = \langle x \rangle_0 + \langle p \rangle_0 \cdot t$$
Relevant Equations

\[
\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}
\]

\[
\int_0^\infty \sqrt{r} e^{-ar}dr = \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}}
\]

\[
\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi(p,t) e^{ipx/\hbar} dp \quad ; \quad \varphi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,t) e^{-ipx/\hbar} dp
\]

\[
\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \frac{1}{2} \frac{\partial}{\partial \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}
\]

\[
\int_{-\infty}^{\infty} e^{i(y-y')x} dx = 2\pi \delta(y - y')
\]

\[
\int_{-\infty}^{\infty} e^{-ax^2 + bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}
\]

\[
\int_{-\infty}^{\infty} x e^{-ax^2 + bx} = \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}
\]

\[
\int_{-\infty}^{\infty} x^2 e^{-ax^2 + bx} = \frac{2a + b}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}
\]

********************

\[
Y_0^0(\theta, \varphi) = \sqrt{\frac{1}{4\pi}}
\]

\[
Y_1^0(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos \theta
\]

\[
Y_1^\pm 1(\theta, \varphi) = \pm \sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin \theta
\]

\[
Y_2^0(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)
\]

\[
Y_2^\pm 1(\theta, \varphi) = \pm \sqrt{\frac{15}{8\pi}} e^{\pm i\varphi} \sin \theta \cos \theta
\]

\[
Y_2^\pm 2(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\varphi} \sin^2 \theta
\]

\[
L_x = \frac{\hbar}{i} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -\frac{\hbar}{i} \left( \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \varphi} \right)
\]

\[
L_y = \frac{\hbar}{i} \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) = \frac{\hbar}{i} \left( \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \varphi} \right)
\]

\[
L_z = \frac{\hbar}{i} \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}
\]

3
\[ L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \]

\[ L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \]

\[ L_z Y_l^m = m \hbar Y_l^m \]

\[ L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \]

Central potential

\[ H = -\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] + V(r) \]

\[ = -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{2mr^2} + V(r) \]

Radial Equation:

\[ -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \left( V + \frac{\hbar^2}{2mr^2} \ell (\ell + 1) \right) R = ER \]

\[ u = rR \]

\[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} u + \left( V + \frac{\hbar^2}{2mr^2} \ell (\ell + 1) \right) u = Eu \]
Part B

3. Consider a particle of mass $m$ and charge $q$ in the three-dimensional harmonic potential

$$V = \frac{1}{2} m \omega^2 (\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2)$$

a) [4 points] After defining the ladder operators $a_i$ and $a_i^\dagger$ as

$$a_i = \frac{1}{\sqrt{2m\omega\hbar}} (\hat{p}_i - im\omega \hat{x}_i) \quad a_i^\dagger = \frac{1}{\sqrt{2m\omega\hbar}} (\hat{p}_i + im\omega \hat{x}_i)$$

Show that $[a_i, a_j^\dagger] = \delta_{ij}$ and that the Hamiltonian of the harmonic oscillator can be written as

$$\hat{H}_0 = \hbar \omega \sum_{i=1}^{3} (a_i^\dagger a_i + \frac{1}{2})$$.

b) [4 points] What are the three lowest energy levels? What are their degeneracies? List the states associated with each level.

c) [3 points] Show that the angular momentum operator $\hat{L}$ can be written in terms of the raising and lowering operators, as $\hat{L}_i = -i\hbar \epsilon_{ijk} a_j^\dagger a_k$, i.e. $\hat{L}_3 = -i\hbar (a_1^\dagger a_2 - a_2^\dagger a_1)$.

d) [3 points] Show that $\hat{L}$ built in this manner commutes with $\hat{H}_0$.

e) [3 points] For the first excited state, find a common set of eigenstates of $\hat{H}_0$ and $\hat{L}_3$. What are the eigenvalues of $\hat{L}_3$?

f) [8 points] The particle is perturbed by a uniform time-dependent electric field $E(t) = A \exp\{-(t/\tau)^2\}$ directed along the positive $z$ axis (i.e. in the direction of $x_3$). Compute the probability, at first order approximation, of finding the particle in an excited state at time $t = +\infty$, if at the time $t = -\infty$ it was in the ground state. Which levels of the harmonic oscillator will be accessible, at first order approximation, under this perturbation? Can we observe a transition to the second excited state?

4. Two particles, whose magnetic moments are $\mu_1 = aS_1$ and $\mu_2 = bS_2$ respectively, interact with an external magnetic field $B$, as well as with each other. We take the $z$-axis in the direction of the field $B$, so that the Hamiltonian of the system can be written as

$$\hat{H} = a \, S_{1z} \, B + b \, S_{2z} \, B + J \, S_1 \cdot S_2$$.

Consider the case $s_1 = s_2 = 1/2$. Define $J = S_1 + S_2$ the total angular momentum of the system.

a) [5 points] The system can be described either using the basis $|m_1m_2\rangle \equiv |s_1s_2m_1m_2\rangle$ (eigenstates of $S_1^2$, $S_2^2$, $S_{1z}$, and $S_{2z}$) or the basis $|jm_j\rangle \equiv |s_1s_2jm_j\rangle$ (eigenstates of $S_1^2$, $S_2^2$, $J^2$, $J_z$). Find the expression of the states $|jm_j\rangle$ in terms of the appropriate elements of the basis $|m_1m_2\rangle$. This part requires the direct construction of all CG coefficients, namely to prove that

$$|1, 1\rangle = |+, +\rangle, \quad |1, 0\rangle = \frac{1}{\sqrt{2}} \left( |+, -\rangle + |-, +\rangle \right), \quad |0, 0\rangle = \frac{1}{\sqrt{2}} \left( |+, -\rangle - |-, +\rangle \right), \quad |1, -1\rangle = |-, -\rangle$$

(where we adopted the usual simplified notation $|+, +\rangle \equiv |+1/2, +1/2\rangle$, etc...).

b) [20 points] Find the exact energy eigenvalues of the system and sketch the spectrum as a function of the magnetic field. Comment on the degeneracy.
Relevant Formulas for Part B

One-Dimensional Harmonic oscillator:

\[
\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2
\]

\[
\hat{a} = \frac{1}{\sqrt{2m\hbar}} (\hat{p} - im\omega\hat{x})
\]

\[
\hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}} (\hat{p} + im\omega\hat{x})
\]

\[
[\hat{a}, \hat{a}^\dagger] = 1
\]

\[
[\hat{H}_0, \hat{a}^\dagger] = \hbar\omega \hat{a}^\dagger
\]

\[
\langle n-1|\hat{a}|n\rangle = \sqrt{n}
\]

\[
\langle n|\hat{a}^\dagger|n-1\rangle = \sqrt{n}
\]

Time-dependent Perturbation Theory (first order transition probability):

\[
P_{k\leftarrow s} = \frac{1}{\hbar^2} \left| \int_{t_0}^{t} dt' \langle k|V(t')|s\rangle e^{i\omega_{ks}t'} \right|^2
\]

where \(\omega_{ks} = \frac{E_k - E_s}{\hbar}\)

Angular Momentum Operators (\(\hbar = 1\)):

\[
[J_x, J_y] = iJ_z \quad ([J_i, J_j] = i\epsilon_{ijk}J_k) \quad [J^2, J_i] = 0
\]

\[
J_z|j \, m\rangle = m|j \, m\rangle, \quad J^2|j \, m\rangle = j(j+1)|j \, m\rangle
\]

\[
J_\pm = J_x \pm iJ_y \quad J_\pm|j \, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} \, |j \, m \pm 1\rangle
\]
THE CITY UNIVERSITY OF NEW YORK  
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics  
January 20, 2016

Instructions:
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PART A  
1. Parts a, b, c, and d are independent of each other.  

(a) [6] The equation of continuity in quantum mechanics for real potentials is given by  

\[ \frac{\partial}{\partial t} |\psi(r,t)|^2 + \nabla \cdot J = 0 \]  

where \( J \) is the quantum mechanical current. Suppose the potential is complex, \( V = V_r + iV_i \). Derive the equation of continuity for this case.  

(b) [6] Consider the following one dimensional wave function  

\[ \psi(x) = xe^{-\alpha x/2} \quad 0 \leq x \]  

where \( \alpha \) is a positive real number. Calculate the uncertainty product. The uncertainty product is the product of the standard deviation of position times the standard deviation of momentum. Of course your answer should come out to be greater than \( \hbar / 2 \).  

(c) [6] Suppose we have an Hermitian operator \( A \) that commutes with the angular momentum operator in the \( z \) direction \( L_z \). Evaluate  

\[ \left\langle Y^m_\ell | A | Y^{m'}_{\ell'} \right\rangle \]  

where \( Y^m_\ell(\theta, \varphi) \) are the usual spherical harmonics. Express your answer in terms of \( \left\langle Y^m_\ell \right\rangle \).  

(d) [7] Suppose we have a wave function and we know that the potential is a harmonic oscillator potential. When we measure the total energy we find that the only values are \( \hbar \omega / 2 \) or \( 3\hbar \omega / 2 \) with relative frequency of 1/5 and 4/5. Also a measurement of the position gives an expectation value of zero. What is the wave function? Hint: take the wave function to be \( \psi(x) = A(u_0(x) + \alpha u_1(x)) \) and impose the stated conditions. You can take \( A \) to be real but \( \alpha \) may be complex. The matrix elements of \( x \) are given in the formulas sheet.
2. Parts a, b, and c are independent of each other

(a) [6] A particle in a box of length $L$ has the following wave function

$$\psi(x) = 1 \quad 0 \leq x \leq L/2$$

and zero for $L/2 \leq x \leq L$. Normalize the wave function and calculate the probability that a measurement of the energy will give a value of $E = \frac{\pi^2 \hbar^2}{2mL^2}$.

(b) [6] In classical mechanics we have for the force, $F = \frac{dp}{dt}$. Suppose that in quantum mechanics we define the force operator by

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

where $\mathbf{p}$ is the quantum mechanical momentum operator. Calculate the expected value of force for a wave function that is an eigenfunction of the Hamiltonian, that is, for a stationary state.

(c) [13] Consider the following 3 dimensional wave function

$$u_{nlm} = r^2 e^{-\alpha r} Y^m_l(\theta, \phi)$$

where $\alpha$ is a real positive number. Determine whether there are values of $\alpha, l, m$ that makes this wave function an eigenfunction of the Hydrogen atom Hamiltonian. If there are values list them all, determine $n$, the principle quantum number, and the corresponding energy. You must show your work.
**Relevant Equations**

\[
\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* = \nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*)
\]

\[
\int_0^\infty x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}}
\]

\[
\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(p,t) e^{ipx/\hbar} \, dp \quad ; \quad \varphi(p,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t) e^{-ipx/\hbar} \, dp
\]

\[
\int_{-\infty}^{\infty} e^{iyx} \, dx = 2\pi \delta(y)
\]

\[
\int_{-\infty}^{\infty} e^{-ax^2 + bx} = \sqrt{\frac{\pi}{a} e^{b^2/(4a)}}
\]

\[
\int_{-\infty}^{\infty} xe^{-ax^2 + bx} = \frac{b}{2a} \sqrt{\frac{\pi}{a} e^{b^2/(4a)}}
\]

\[
\int_{-\infty}^{\infty} x^2 e^{-ax^2 + bx} = \frac{2a + b}{4a^2} \sqrt{\frac{\pi}{a} e^{b^2/(4a)}}
\]

**L^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$$

\[
L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \quad \text{and} \quad L_z Y_{l}^{m} = m \hbar Y_{l}^{m}
\]

\[
L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \quad \text{and} \quad L^2 Y_{l}^{m} = l(l+1) \hbar^2 Y_{l}^{m}
\]

**Central potential**

\[
H = -\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] + V(r)
\]

\[
= -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{2mr^2} + V(r)
\]

**Radial Equation:**

\[
-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) R = ER
\]

\[
R = u
\]

\[
-\frac{\hbar^2}{2mdr^2} u + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) u = Eu
\]
Hydrogen atom:

\[ V(r) = -\frac{e^2}{r} \]  

\[ -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \left( -\frac{e^2}{r} + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) R = ER \]  

\[ a_0 = \frac{\hbar^2}{me^2} \]  

\[ E_n = -\frac{e^2}{2a_0 n^2} = -\frac{me^4}{2\hbar^2 n^2} \]  

\[ l = 0, 1, ..., n - 1 \]  

One dimensional harmonic oscillator

\[ \left( \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 \right) u_n(x) = E_n u_n(x) \]

\[ u_n(x) = N_n e^{-\alpha^2 x^2/2} H_n(\alpha x) \]

\[ \alpha = \sqrt{\frac{m\omega}{\hbar}} = \left( \frac{mk}{\hbar^2} \right)^{1/4} \quad \omega = \sqrt{\frac{k}{m}} \quad N_n = \left( \frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} \]

where

\[ E_n = (n + \frac{1}{2}) \hbar \omega \quad \alpha = \sqrt{\frac{m\omega}{\hbar}} \]

and

\[ u_n(x) = N_n H_n(\alpha x) e^{-\frac{1}{2} \alpha^2 x^2} \]

\[ u_0(x) = \left( \frac{\alpha^2}{\pi} \right)^{1/4} e^{-\alpha^2 x^2/2} \]

\[ u_1(x) = \sqrt{2\pi} \left( \frac{\alpha^2}{\pi} \right)^{3/4} x e^{-\alpha^2 x^2/2} \]

\[ \int u_n(x) x u_m(x) \frac{1}{\alpha} \left[ \sqrt{\frac{n+1}{2}} \delta_{m,n+1} + \sqrt{\frac{n}{2}} \delta_{m,n-1} \right] \]  

\[ \int u_n(x) p u_m(x) = -i\hbar \left[ \sqrt{\frac{n+1}{2}} \delta_{m,n+1} - \sqrt{\frac{n}{2}} \delta_{m,n-1} \right] \]

For a particle in a box of length \( L \) the eigenvalues and normalized eigenfunctions are

\[ E_n = \frac{\pi^2 \hbar^2}{2mL^2} n^2 \quad ; \quad u_n(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right) \]
Part B

3. We want to combine two angular momenta \( L \) and \( S \), and describe the state of the system in the basis of eigenstates of the operators \( J^2 \) and \( J_z \) (where \( J = L + S \)).

a) [5 points] Assuming that \( L \) and \( S \) commute, explain why \( \{ L^2, S^2, J^2, J_z \} \) and \( \{ L^2, S^2, L_z, S_z \} \) are two alternative “good” sets of compatible observables, while this is not the case for \( \{ L_z, S_z, J^2, J_z \} \).

Define the Clebsch-Gordan (CG) coefficients.

b) For the case \( l = 1 \) and \( s = 1/2 \):

b.1 [2 points] How many states belong to each of the two orthonormal and complete sets \( \{|m_l m_s\rangle\} \) and \( \{|j m_j\rangle\} \)? List all the elements of each set.

A quantum system is prepared in the state

\[
|\psi\rangle = \sqrt{\frac{1}{3}} |j = \frac{3}{2}, m_j = \frac{3}{2}\rangle + \sqrt{\frac{2}{3}} |j = \frac{1}{2}, m_j = \frac{1}{2}\rangle
\]

b.2 [10 points] Find the expression of the states \( |j m_j\rangle \) which appear in \( |\psi\rangle \) in terms of the appropriate elements of the basis \( \{|m_l m_s\rangle\} \). This part requires the direct construction of all CG coefficients relevant to the solution of the problem.

b.3 [3 points] What is the probability of measuring \( m_s = \frac{1}{2} \) in the state \( |\psi\rangle \)? What about \( m_s = -\frac{1}{2} \)?

b.4 [3 points] What about a measurement of \( m_l \)? List the probabilities of finding \( m_l = +1, 0, -1 \).

b.5 [2 points] What is the probability of a combined measurement \( m_l = 1 \) and \( m_s = \frac{1}{2} \) in \( |\psi\rangle \)?

4. This problem is formed by two independent parts (4a and 4b).

4a. Consider a three-state quantum system described by the Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{H}_p \) where

\[
\hat{H}_0 = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \hat{H}_p = b \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix},
\]

and \( b << a \). We want to apply non-degenerate perturbation theory to study the full Hamiltonian \( \hat{H} \).

1) [10 points] After writing down the eigenstates and eigenvalues of \( \hat{H}_0 \), compute the first- and second-order corrections to the energy levels due to \( \hat{H}_p \).

2) [5 points] Compute the exact eigenvalues of \( \hat{H} \). Check that, by expanding this result in \( \epsilon = \frac{b}{a} \), you can reproduce your results from part b).

4b. [10 points] Use the WKB approximation to find the allowed energy levels for the one-dimensional harmonic oscillator.
Relevant Formulas for Part B

Angular Momentum Operators ($\hbar = 1$):

$$[J_x, J_y] = iJ_z \quad ([J_i, J_j] = i\epsilon_{ijk}J_k) \quad [J^2, J_i] = 0$$

$$J_z |j m\rangle = m |j m\rangle, \quad J^2 |j m\rangle = j(j+1) |j m\rangle$$

$$J_\pm = J_x \pm iJ_y \quad J_\pm |j m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j m \pm 1\rangle$$

Time-independent Perturbation Theory:

$$E^1_n = \langle n | \hat{H}_p | n \rangle; \quad E^n_i = \langle p^0_n | \hat{H}_p | p^{i-1}_n \rangle; \quad |p^1_n\rangle = \sum_{k \neq n} \frac{\langle k | \hat{H}_p | n \rangle}{E^0_n - E^0_k} |k\rangle$$
THE CITY UNIVERSITY OF NEW YORK
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics
June 15, 2016

Instructions:
a) PUT YOUR IDENTIFICATION NUMBER ON EACH PAGE.
b) There are two parts to this exam, parts A and B. Do two questions but you must chose one from each part. Each question is worth 50 points.
c) Start each problem on a new page.
d) Indicate clearly which two problems you choose to solve.

1. Parts a, b, c, and are independent of each other.

(a) [16] Consider the spin operator
\[ A = 3\sigma_y + 4\sigma_z \]
where \( \beta \) is a real number. Find the eigenvalues and the normalized eigenvectors of \( A \).

(b) [16] Consider the following one dimensional time dependent wave function in momentum space
\[ \varphi(p, t) = A e^{-\eta(t)p^2/2} \]
where \( \eta(t) \) is a complex function of time and also satisfies
\[ \eta(t) + \eta^*(t) = 2c \]
where \( c \) is a positive constant. Suppose you are told this wave function satisfies the Schrödinger equation for a free particle. Normalize the wave function and calculate \( \eta(t) \) by applying the Schrödinger equation in momentum space. Further, calculate the position wave function.

(c) [18] Consider the central potential
\[ V(r) = -\alpha e^{-\beta r} \]
where \( \alpha \) and \( \beta \) are positive constants. The standard equation for \( U(r) = rR(r) \) for \( \ell = 0 \) case is
\[ \frac{\hbar^2}{2m} \frac{d^2}{dr^2} U + V U = EU \]
Make the substitution
\[ x = Ae^{-Br} \]
and express \( U \) as a Bessel function. To make your work easier you may use that
\[ \frac{d}{dr} = -Bx \frac{d}{dx} \]
\[ \frac{d^2}{dr^2} = B^2 \left( x \frac{d}{dx} + x^2 \frac{d^2}{dx^2} \right) \]
\[ e^{-\beta r} = \left( \frac{x}{A} \right)^{\beta/B} \]
2. Parts a, b, and c are independent of each other

(a) [20] Consider a particle in a infinite cylinder where the potential is

\[ V(\rho) = \begin{cases} 
0 & \text{if } \rho < R, \\
\infty & \text{if } \rho > R. 
\end{cases} \]

Assume that the eigenfunctions in cylindrical coordinates are of the following form

\[ u(\rho, \varphi, z) = \frac{e^{ipz/\hbar}}{\sqrt{2\pi\hbar}} \frac{e^{im\varphi}}{\sqrt{2\pi}} R(\rho) \]

Find the equation that \( R(\rho) \) must satisfy and express it in terms of Bessel functions. To express the solution in terms of Bessel functions let \( x = \alpha \rho \) and find \( \alpha \) so that the \( R \) becomes a Bessel function. Also explain how you would find the eigenvalues.

(b) [14] For a hydrogenic atom with nuclear charge \( Z \) the ground state wave function is

\[ u(r, \theta, \varphi) = \frac{1}{\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0} \]

What is the probability distribution for the electron as a function of \( r \). Find where the probability distribution is a maximum. Then calculate \( \langle r \rangle \) and compare.

(c) [16] Consider the following three dimensional Hamiltonian

\[ H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \alpha L_z \]

where \( \alpha \) is a real number and where \( L_z \) is the angular momentum operator in the \( z \) direction

\[ L_z = xp_y - yp_x \]

Using the Heisenberg equation of motion write the equations that can be used to obtain \( x \) and \( y \) as functions of time. Do not solve them but be sure to have as many equations as there are unknowns. Remember that \([x, p_x^2] = 2i\hbar p_x\), etc.
Equations for part A

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

cylindrical coordinates:

\[ \nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \]

Central potential

\[ H = -\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] + V(r) \]

Radial Equation:

\[ -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) R = ER \]

\[ u = rR \]

\[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} u + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) u = Eu \]

Bessel function that is finite at zero:

\[ \frac{d^2}{dx^2} J_\lambda + \frac{1}{x} \frac{d}{dx} J_\lambda + \left( 1 - \frac{\lambda^2}{x^2} \right) J_\lambda = 0 \]

Integrals

\[ \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \]

\[ \psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^\infty \phi(p, t) e^{ipx/\hbar} dp \]

\[ \varphi(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^\infty (x, t) e^{-ipx/\hbar} dp \]

\[ \int_{-\infty}^\infty e^{i(y-y')x} dx = 2\pi \delta(y - y') \]

\[ \int_{-\infty}^\infty e^{-ax^2 + bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^\infty xe^{-ax^2 + bx} = \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^\infty x^2 e^{-ax^2 + bx} = \frac{2a + b}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]
Part B

Problem 3. [50 points] Consider an isotropic harmonic oscillator in two dimensions, whose Hamiltonian is given by

\[ H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m \omega^2 x^2 + \frac{1}{2} m \omega^2 y^2 \]

a) [5 points] What are the energy levels of the three lowest states (ground state, first and second excited states)? What are their degeneracies?

We now apply a perturbation \( V = \beta m \omega^2 x y \) where \( \beta \) is a real number.

b) [5 points] What are the conditions that the parameter \( \beta \) should satisfy in order for perturbation theory to be applicable? (i.e. clarify how “small” \( \beta \) should be).

c) [20 points] Compute the energy levels at first order in perturbation theory for each of the three lowest states. Is the degeneracy removed by \( V \)?

d) [5 points] For the first excited state compute the eigenstates corresponding to the energy levels that you found in part c).

e) [15 points] Solve the \( H = H_0 + V \) problem exactly and compare with the results obtained that you obtained in part b). Hint: the change of variables \( q_x = (x + y)/\sqrt{2}, q_y = (x - y)/\sqrt{2} \) leads to a simpler expression for the Hamiltonian.

Problem 4. [50 points] This Problem is made of two independent parts.

4a. The operators \( S_1 \) and \( S_2 \) are associated with intrinsic angular momenta \( s_1 = 1 \) and \( s_2 = 1 \). The operator \( J \) is defined as the sum of \( S_1 \) and \( S_2 \), namely \( J = S_1 + S_2 \).

a) [5 points] How many states belong to each of the two o.n.c. sets \( \{|s_1 s_2 m_1 m_2 \rangle\} = \{|m_1, m_2 \rangle\} \) and \( \{|s_1 s_2 j m_j \rangle\} = \{|j, m_j \rangle\} \)? List all the elements of each set.

b) [10 points] Find the expression of the states \( \{|2, m_j \rangle\}, \) for \( m_j = 2, \ldots, -2 \) in terms of the appropriate elements of the basis \( \{|m_l, m_s \rangle\} \). This part requires the direct construction of all CG coefficients relevant to the solution of the problem.

c) [10 points] Compute the expectation value \( < \hat{A} >_\phi \) of the operator \( \hat{A} = a \ S_{1,z} + b \ S_{2,z} + c \ S_1 \cdot S_2 \) in the state

\[ |\phi\rangle = \sqrt{\frac{1}{2}} \left[ |j = 2, m_j = 2 \rangle - |j = 2, m_j = 1 \rangle \right] \]

4b. An angular-momentum eigenstate \( |j = 3, m = 3 \rangle \) is rotated by an infinitesimal angle \( \epsilon \) about the y-axis.

a) [15 points] Compute the probability for the new rotated state to be found in the original state up to terms of order \( \epsilon^2 \). Compute the probability for the new rotated state to be found in a different state (again up to \( \epsilon^2 \)). Verify that all the probabilities (up to order \( \epsilon^2 \)) sum up to one.

b) [5 points] At which order in \( \epsilon \), if any, would a transition to the state \( |j = 2 m = 2 \rangle \) be allowed? Explain.

c) [5 points] Repeat part a) for an rotation about the z-axis.
Relevant Formulas for Part B

One-Dimensional Harmonic oscillator:

\[ \hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \]
\[ \hat{a} = \frac{1}{\sqrt{2m\hbar}} (\hat{p} - im\omega \hat{x}) \]
\[ \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar}} (\hat{p} + im\omega \hat{x}) \]
\[ [\hat{a}, \hat{a}^\dagger] = 1 \]
\[ [\hat{H}_0, \hat{a}^\dagger] = \hbar \omega \hat{a}^\dagger \]
\[ \langle n - 1 | \hat{a} | n \rangle = \sqrt{n} \]
\[ \langle n | \hat{a}^\dagger | n - 1 \rangle = \sqrt{n} \]

Time-independent Perturbation Theory:

\[ E_n^1 = \langle n | \hat{H}_p | n \rangle ; \quad E_n^i = \langle p_n^0 | \hat{H}_p | p_n^{i-1} \rangle ; \quad | p_n^1 \rangle = \sum_{k \neq n} \frac{(k|\hat{H}_p|n)}{E_n^0 - E_k^0} |k\rangle \]

Degenerate case:

\[ \det \left[ E_n^1 - \hat{H}_p^0 \right] = 0, \]

where \( \hat{H}_p^0 = \langle n_i | \hat{H}_p | n_j \rangle \) and \{\( |n_i\rangle \)\} is the subspace corresponding to the degenerate eigenvalue \( E_n^0 \) of \( \hat{H}_0 \).

Angular Momentum Operators (\( \hbar = 1 \)):

\[ [J_z, J_y] = iJ_x \quad ([J_i, J_j] = i\epsilon_{ijk}J_k) \quad [J^2, J_i] = 0 \]
\[ J_z |j m\rangle = m |j m\rangle , \quad J^2 |j m\rangle = j(j + 1) |j m\rangle \]
\[ J_{\pm} = J_x \pm iJ_y \quad J_{\pm} |j m\rangle = \sqrt{j(j + 1) - m(m \pm 1)} |j m \pm 1\rangle \]

Infinitesimal Rotation Operators (angle \( \theta \) about direction \( \hat{n} \), \( \hbar = 1 \)).

\[ R(\hat{n}, \theta) = \exp \left( -i\theta \hat{J} \cdot \hat{n} \right) \]
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PART A

1. Consider the following wave function

\[ u(r, \theta, \phi) = N r e^{-\alpha r/2} \cos \theta \]  

(a) [10] Normalize the wave function and calculate the standard deviation of the radial position.

(b) [15] Calculate \( \nabla^2 u(r, \theta, \phi) \)

(c) [5] Calculate the expected value of kinetic energy

(d) [15] Consider the standard central potential Hamiltonian,

\[ H = -\frac{\hbar^2}{2m} \nabla^2 + V(r) \]

Does there exist a potential, \( V(r) \), so that \( u \) as given by Eq. (1) is an eigenfunction of \( H \)? The answer is yes. Find \( V(r) \) and the corresponding energy eigenvalue

(e) [5] For the potential you found in part d) calculate its expected value

2. Parts a) and b) of this problem are independent of each other

(a) Suppose that for the operator \( A \) given by

\[ A = p - (\beta + i\lambda)x \]

the normalized wave function \( \psi(x) \) obeys

\[ A\psi(x) = (b - i\lambda a)\psi(x) \]  

In these equations \( a, b, \beta, \lambda \) are real numbers.

i. [10] Find \( \langle p \rangle \) and \( \langle x \rangle \). Hint: calculate \( \langle A \rangle \)

ii. [15] Using (2) find \( \psi(x) \) and normalize it
iii. [5] Using the result found in ii) calculate directly $\langle p \rangle$ and $\langle x \rangle$ and verify what you obtained in i)

(b) Suppose $u_1$ and $u_2$ are eigenfunctions of a Hermitian Hamiltonian with corresponding eigenvalues $E_1$ and $E_2$. Consider the wave function $\psi(x, 0)$ at time zero.

$$\psi(x, 0) = a u_1(x) + b u_2(x)$$

where $a$ and $b$ are real numbers. Also $E_1 \neq E_2$

i. [5] Normalize the wave functions and write $\psi(x, t)$

ii. [15] Suppose we have an Hermitian operator $A$ which has the following properties

$$A(u_1 + u_2) = \alpha_1(u_1 + u_2)$$
$$A(u_1 - u_2) = \alpha_2(u_1 - u_2)$$

where $\alpha_1, \alpha_2$ are real. Calculate the expected value of $A$ at time $t$, that is $\langle A \rangle_t$. Since $A$ is hermitian the expected value is real – hence make sure that all the terms in your answer are real.
Equations PART A

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \]

Central potential

\[ H = -\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] + V(r) \]

Radial Equation:

\[ -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) R = ER \]

\[ u = rR \]

\[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} u + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) u = Eu \]

Integrals

\[ \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \]

\[ \psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \varphi(p, t) e^{ipx/\hbar} dp \quad ; \quad \varphi(p, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \psi(x, t) e^{-ipx/\hbar} dp \]

\[ \int_{-\infty}^\infty e^{i(y-y')x} dx = 2\pi \delta(y - y') \]

\[ \int_{-\infty}^\infty \frac{\sqrt{\pi}}{a} e^{b^2/(4a)} \]

\[ \int_{-\infty}^\infty xe^{-ax^2+bx} = \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^\infty x^2 e^{-ax^2+bx} = \frac{2a+b}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]
Quantum Mechanics – Part B

3. We want to combine two angular momenta \( L \) and \( S \), and describe the state of the system in the basis of eigenstates of the operators \( J^2 \) and \( J_z \) (where \( J = L + S \)).

a) [5 points] Assuming that \( L \) and \( S \) commute, show explicitly that the operators \( J^2 \) and \( L_z \) do not commute. What are the consequences of this fact?

b) [5 points] Define the Clebsch-Gordan (CG) coefficients that link the complete sets of states \( \{ |m_l m_s \rangle \} \) and \( \{ |j m_j \rangle \} \). What are the allowed values of the quantum numbers \( j \) and \( m_j \)? (given the values of \( l \) and \( s \)). No proof needed.

c) [5 points] Show that all GC coefficients \( \langle m_l m_s | j m_j \rangle \) are identically zero, unless \( m_j = m_l + m_s \).

For the specific case \( l = 4 \) and \( s = 2 \):

d) [20 points] Find the expression of the following states in terms of the appropriate elements of the basis \( \{ |m_l m_s \rangle \} \). This part requires the direct construction of all CG coefficients relevant to the solution of the problem.

\[ \begin{align*}
\text{d.1} : & \quad |j = 6 \ m_j = 6 \rangle \\
\text{d.2} : & \quad |j = 6 \ m_j = 4 \rangle \\
\text{d.3} : & \quad |j = 5 \ m_j = 5 \rangle \\
\text{d.4} : & \quad |j = 1 \ m_j = 1 \rangle \\
\end{align*} \]

e) A quantum system is prepared in the state

\[ |\psi\rangle = \sqrt{\frac{1}{3}} |j = 6 \ m_j = 6 \rangle + \sqrt{\frac{2}{3}} |j = 5 \ m_j = 5 \rangle. \]

\[ \text{e.1 : [10 points]} \] List the probabilities of all possible outcomes of a measurement of \( m_s \) performed in the quantum state \( |\psi\rangle \) (and check that their sum is consistent).

\[ \text{e.2 : [5 points]} \] What about a measurement of \( m_j \)?

4. Using perturbation theory, we want to study the one-dimensional “slightly relativistic” harmonic oscillator, whose Hamiltonian can be written as

\[ \hat{H} = \hat{H}_0 + \hat{H}_{rel} \quad \text{where} \quad \hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \]

Define the operators \( a \) and \( a^\dagger \) as

\[ \begin{align*}
a & = \frac{1}{\sqrt{2m\omega\hbar}} (\hat{p} - i m \omega \hat{x}) \quad a^\dagger = \frac{1}{\sqrt{2m\omega\hbar}} (\hat{p} + i m \omega \hat{x}) \quad [a, a^\dagger] = 1
\end{align*} \]

a) [5 points] Show that the hamiltonian of the harmonic oscillator can be written as \( \hat{H}_0 = \hbar \omega (a^\dagger a + \frac{1}{2}) \).
b) [5 points] Show that
\[ \hat{H}_{\text{rel}} = -\frac{\hat{p}^4}{8mc^2} \]
by expanding the relativistic expression for the kinetic energy \( T = \sqrt{\hat{p}^2c^2 + m^2c^4} - mc^2 \) in the limit of “slightly relativistic” velocity.

c) [20 points] Using the properties of the ladder operators \( a \) and \( a^\dagger \), find the expression for \( \langle m|\hat{p}^4|0\rangle \).

d) [10 points] Calculate the leading non-vanishing energy shift of the ground state due to this relativistic perturbation.

e) [10 points] Calculate the leading corrections to the ground state eigenvector \( |0\rangle \).

**Relevant Formulas for Part B**

Time-independent Perturbation Theory:

\[ E_n^1 = \langle n|\hat{H}_p|n\rangle \quad , \quad E_n^i = \langle n|\hat{H}_p|n^{i-1}\rangle \]
\[ |p_n^1\rangle = \sum_{k\neq n} \frac{\langle k|\hat{H}_p|n\rangle}{E_n^0 - E_k^0}|k\rangle \]

Angular Momentum Operators (\( \hbar = 1 \)):

\[ [J_x, J_y] = iJ_z \quad ([J_i, J_j] = i\epsilon_{ijk}J_k) \quad [J^2, J_i] = 0 \]
\[ J_z|m \rangle = m|m \rangle , \quad J^2|m \rangle = j(j + 1)|m \rangle \]
\[ J_\pm = J_x \pm iJ_y \quad J_\pm|m \rangle = \sqrt{j(j + 1) - m(m \pm 1)}|j \pm 1 \rangle \]

Harmonic oscillator:

\[ \hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \]
\[ a = \frac{1}{\sqrt{2m\hbar}}(p - im\omega x) \quad , \quad a^\dagger = \frac{1}{\sqrt{2m\hbar}}(p + im\omega x) \]
\[ [a, a^\dagger] = 1 \quad , \quad [\hat{H}_0, a^\dagger] = \hbar\omega a^\dagger \]
\[ \langle n - 1|a|n\rangle = \sqrt{n} \quad , \quad \langle n|a^\dagger|n - 1\rangle = \sqrt{n} \]
THE CITY UNIVERSITY OF NEW YORK
First Examination for Ph.D. Candidates in Physics – Quantum Mechanics
June 19, 2017

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d) Indicate clearly which two problems you choose to solve.

PART A

1. Consider an initial wave function, \( \psi(x, 0) \), and suppose that it is evolved in two different ways: as a free particle and with a linear potential,

\[
i \hbar \frac{\partial}{\partial t} \psi_f(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_f(x, t) \\
i \hbar \frac{\partial}{\partial t} \psi(x, t) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - Fx \right) \psi(x, t)
\]

The aim of this problem is to show that there is a relation between \( \psi(x, t) \) and \( \psi_f(x, t) \). The relation is of the form

\[
\psi(x, t) = e^{iA(x, t)} \psi_f(x - Ft^2/2m, t)
\]  

(1)

Show this and explicitly obtain \( A(x, t) \). The best way to do this problem is to first do it in momentum space following these steps.

(a) [15] First prove that for the linear potential the momentum wave function at time zero \( \varphi(p, t) \) is related to \( \varphi(p, 0) \) by

\[
\varphi(p, t) = \exp \left[ -\frac{i}{\hbar} \left( \frac{p^2}{2m} t - \frac{pFt}{2m} t^2 + \frac{F^2 t^3}{6m} \right) \right] \varphi(p - Ft, 0)
\]

Do this by showing that \( \varphi(p, t) \) satisfies Schrödinger's equation in momentum space.

(b) [15] Express \( \varphi(p, t) \) in terms of \( \varphi_f(p, t) \) where \( \varphi_f(p, t) \) is the momentum wave functions for the free particle case. (Remember we are assuming that both cases have the same initial wave function.)

(c) [20] Using your result from part b) show Eq. (1) and obtain \( A(x, t) \).
2. Parts (a) and (b) of this problem are independent of each other

(a) Consider the eigenvalue problem for the Hamiltonian in two dimensions expressed in polar coordinates and where the potential is a radial potential $V(\rho)$. The Laplacian in polar coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2}$$

i. [15] Use separation of variables to find the equations for the radial and polar part.
That is take the wave function to be of the from $\psi(\rho, \varphi) = R(\rho)\Phi(\varphi)$ and find the equations that $R(\rho)$ and $\Phi(\varphi)$ satisfy. For the solution of $\Phi(\varphi)$ take $\Phi(\varphi) = e^{im\varphi}$

ii. [15] Define

$$u(\rho) = \rho^\alpha R(\rho)$$

Find the equation that $u(\rho)$ satisfies and then determine for what value of $\alpha$ the equation for $u(\rho)$ has the same form as a one dimensional Schrödinger equation. That is, of the form

$$-\frac{\hbar^2}{2m} u'' + V_{\text{eff}}(\rho)u = Eu$$

iii. [10] What is the effective potential, $V_{\text{eff}}(\rho)$.

(b) [10] Consider the energy eigenfunctions for a one dimensional problem

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] u(x) = Eu(x)$$

where the eigenfunctions are normalized. The Hamiltonian is a function of the mass $m$ and obviously the eigenfunctions will also be functions of the mass. The potential is indepednt of the mass. Show that

$$\frac{\partial}{\partial m} E < 0$$

Hint: start with

$$E = \int u^*(x)H u(x) dx$$

Equations PART A

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} u(x) e^{-ipx/\hbar} dx$$
Part B

Problem 3.
Let's consider the problem of scattering in a potential with spherical symmetry.

a) [20 pts] Show that, in the Born approximation and for a spherically symmetric potential, the general expression for the scattering amplitude \( f(k, k') = -\frac{(2\pi)^2}{4\pi} \frac{2m}{\hbar^2} \langle k' | V | \psi^{(+)} \rangle \) reduces to

\[
f_B = -\frac{2m}{\hbar^2 q} \int_0^\infty dr \ r \ V(r) \sin qr
\]

where \( q = \tilde{k} - \tilde{k}' \) and \( q = 2k \sin \theta/2 \).

b) [25 pts] Compute in the Born approximation the differential and the total cross section for the Gaussian potential

\[
V(r) = V_0 \exp \left( -\frac{r^2}{2a^2} \right)
\]

c) [5 pts] Show that the differential cross sections of part b) becomes isotropic in the limit \( ka << 1 \).

Problem 4. We want to combine two angular momenta \( L \) and \( S \), with \( l = 2 \) and \( s = 1 \) respectively, and describe the state of the system in the basis of eigenstates of the operators \( J^2 \) and \( J_z \) (where \( J = L + S \)).

a) [10 pts] How many states belong to each of the two orthonormal and complete sets \( \{ |m_i \ m_s \} \) and \( \{ |j \ m_j \} \)? List them.

b) [20 pts] Find the expression of the states \( |j = 3 \ m_j = 0 \rangle \) and \( |j = 2 \ m_j = 2 \rangle \) in terms of the appropriate elements of the basis \( \{ |m_i \ m_s \} \). This part requires the direct construction of all Clebsch-Gordan coefficients relevant to the solution of the problem.

c) [10 pts] Using the components of two spherical tensors, \( A \) of rank one and \( B \) of rank two, we can construct all the components spherical tensor \( C \) of rank three. Build explicitly the components \( C_3^{(3)} \) and \( C_0^{(3)} \) of \( C \) as a function of the components of \( A \) and \( B \).

d) [10 pts] Assume we want to compute the matrix elements \( \langle nlm | C_3^{(3)} | n'l'm' \rangle \) for the \( n = 3 \) states of the hydrogen atom. Which ones of the matrix elements are not trivially equal to zero, according to the selection rules? (You do not need to compute their values!)
Relevant Formulas for Part B

Angular Momentum Operators \((\hbar = 1)\):

\[
[J_x, J_y] = iJ_z \quad ([J_x, J_z] = i\epsilon_{ijk} J_k) \quad [J^2, J_i] = 0
\]

\[
J_x |j \, m\rangle = m |j \, m\rangle, \quad J^2 |j \, m\rangle = j(j + 1) |j \, m\rangle
\]

\[
J_\pm = J_x \pm iJ_y \quad J_\pm |j \, m\rangle = \sqrt{j(j + 1) - m(m \pm 1)} |j \, m \pm 1\rangle
\]

Products of Spherical Tensors and Wigner-Eckart Theorem:

\[
T_q^{(k)} = \sum_{q_1} \sum_{q_2} \langle k_1 \, k_2, \, q_1 \, q_2 | k_1 \, k_2, \, k \, q \rangle X_{q_1}^{(k_1)} Y_{q_2}^{(k_2)} \quad \text{where} \quad k = k_1 + k_2
\]

\[
\langle \alpha' j' m' | T_q^{(k)} | \alpha j m \rangle = \langle j \, k_1 \, k \, q \mid j \, k_1 \, k \, q \rangle \langle \alpha' j' \mid T_q^{(k)} \mid \alpha j \rangle
\]

Useful Integrals:

\[
\int_{-\infty}^{+\infty} dx \ x \ \exp(i\beta x - \alpha x^2) = \sqrt{\frac{\pi \, i\beta}{\alpha^3 \ 2}} \ \exp\left(-\frac{\beta^2}{4\alpha}\right)
\]
There are two parts to this exam, parts A and B. Do two questions but you must chose one from each part. Each question is worth 50 points. Start each problem on a new page.

**PART A**

1. Parts a) and b) are independent of each other.

   (a) Consider the Hamiltonian
   \[ H = \frac{p^2}{2m} - Fx + Gp \]

   i. [18] Using Heisenberg’s equation of motion write and solve the equations for the operators \( p(t) \) and \( x(t) \). That is, find \( p(t) \) and \( x(t) \) in terms of \( p(0) \) and \( x(0) \).

   ii. [18] Suppose the state function at time zero is \( (x, 0) = Ae^{-\alpha x^2/2+i\beta x^2/2+ikx} \). Calculate the correlation coefficients in position, \( C(t) \), defined by
   \[ C(t) = \frac{1}{2} \langle x(t)x(0) + x(0)x(t) \rangle \]

   (b) [14] For the time dependent Schrödinger equation, the equation of continuity is
   \[ \frac{\partial}{\partial t} \rho(x,t) = -\frac{\partial}{\partial x} j(x,t) \]
   where \( \rho(x,t) = |\psi(x,t)|^2 \) and \( j(x,t) \) is the quantum mechanical current. This holds for real potentials. Obtain \( \frac{\partial}{\partial t} \rho(x,t) \) when the potential is complex.
   \( V(x) = V_R(x) + iV_I(x) \)

2. Parts a and b are independent of each other.

   (a) [30] A particle of mass \( m \) is constrained to move between two concentric spheres of radius \( a \) and \( b \) and where the potential is
   \[ V(r) = \begin{cases} 
   \infty & \text{if } r < a, \\
   0 & \text{if } a < r < b, \\
   \infty & \text{if } r > b. 
   \end{cases} \]
   Find the normalized energy eigenfunctions and corresponding energies for the case \( l = 0 \). Make sure you normalize the eigenfunctions – remember this is a 3-D problem. Hint: For \( u(r) \) as given by Eq. (4) of the equation page take
   \[ u(r) = A \sin k(r-a) + B \cos k(r-a) \]

   (b) i. [10] Consider the operator \( D \) given by
   \[ D = \frac{1}{2}(xp + px) = xp - ih/2 \]
   This operator has to do with scaling of position. Calculate \( [D, p] \), \( [D, p^2] \)
   ii. [10] Using the results of part i) show that for a free particle Hamiltonian \( H \)
   \[ [D, H] = 2ihH \]
   and show how \( \langle D(t) \rangle \) evolves in time.
Relevant Formulas

\[ [\mathbf{x}, \mathbf{p}] = i\hbar \]

\[ \dot{\psi}(x) = \frac{\hbar}{2mi} \left( *x \left( \frac{\partial}{\partial x} \psi(x) - \psi(x) \frac{\partial}{\partial x} *x \right) \right) \]

\[ i\hbar \frac{d\mathbf{A}}{dt} = [\mathbf{A}, \mathbf{H}] \]

For central forces:

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi = E\psi \tag{1} \]

\[ = R(r)Y_m^\ell(\theta, \varphi) \tag{2} \]

\[ R(r) \]

\[ -\frac{\hbar^2}{2mr^2} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) R = ER \tag{3} \]

Let

\[ u(r) = rR(r) \tag{4} \]

then

\[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} u + \left( V + \frac{\hbar^2}{2mr^2} \ell(\ell + 1) \right) u = Eu \tag{5} \]

\[ \int_{-\infty}^{\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^{\infty} xe^{-ax^2+bx} = \frac{b}{2a} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int_{-\infty}^{\infty} x^2 e^{-ax^2+bx} = \frac{2a + b}{4a^2} \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \]

\[ \int \sin^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} \]
Part B

Problem 3.
Suppose that the state of a particle of spin $1/2$ can be written, in the basis of $\hat{S}_z$ eigenstates, as

$$|\chi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 + i \2 \end{pmatrix}. $$

3.a) Using the general properties of $\hat{S}_\pm$, $\hat{S}^2$, and $\hat{S}_z$, write explicitly the matrix representation of the operators $\hat{S}^2$, $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ in the basis of eigenstates of $\hat{S}_z$, namely show that for $s = 1/2$:

$$\hat{S}^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

3.b) What are the possible outcomes, and related probabilities, of a measurement of $\hat{S}_z$ in the state $|\chi\rangle$?

3.c) Rewrite $|\chi\rangle$ in the basis of eigenstates of $\hat{S}_z$. What are the probabilities of measuring each one of its eigenvalues? Show that probabilities are correctly normalized.

3.d) Compute the expectation values of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ in the state $|\chi\rangle$.

3.e) What are the possible outcomes in a measurement of $\hat{S}_x^2$ on the state $|\chi\rangle$? With what probability?

3.f) If we first perform a measurement of $\hat{S}_z$ on the state $|\chi\rangle$, followed by a measurement of $\hat{S}_y$, and finally a second measurement of $\hat{S}_z$, what are the possible outcomes of the last measurement and the related probabilities?

Problem 4. A physical system is described by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_p$, where $\hat{H}_p$ is a “small” perturbation to a known problem described by $\hat{H}_0|n\rangle = E^0_n|n\rangle$.

4.a) Derive the general expression for the first order corrections $|p^1_n\rangle$ in perturbation theory to the energy eigenstates and the second order correction $E^2_n$ to the energy eigenvalues, namely show that

$$|p^1_n\rangle = \sum_{k \neq n} \frac{\langle k|\hat{H}_p|n\rangle}{E^0_n - E^0_k} |k\rangle \quad \text{and} \quad E^2_n = \sum_{k \neq n} \frac{\langle k|\hat{H}_p|n\rangle|^2}{E^0_n - E^0_k}. $$

Assume the spectrum $E^0_n$ to be non-degenerate.
Using perturbation theory, we now want to study the energy levels of the one-dimensional anharmonic oscillator. The Hamiltonian of the system can be written as \( \hat{H} = \hat{H}_0 + \hat{H}_3 \), where

\[
\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \quad \hat{H}_3 = \sigma \hat{x}^3.
\]

Define the operators \( \hat{a} \) and \( \hat{a}^\dagger \) as

\[
\hat{a} = \frac{1}{\sqrt{2m\omega}} (\hat{p} - i m \omega \hat{x}) \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega}} (\hat{p} + i m \omega \hat{x})
\]

4.b) Using the properties of the operators \( \hat{a} \) and \( \hat{a}^\dagger \), find the expression for all the non-vanishing matrix elements \( \langle m | \hat{x}^3 | n \rangle \).

4.c) Calculate the first order and second order corrections to the \( n = 2 \) energy level of the harmonic oscillator due to \( \hat{H}_3 \).

4.d) Find the first order corrections to the ground state \( |0\rangle \) of the harmonic oscillator due to \( \hat{H}_3 \).

**Relevant Formulas for Part B**

One-dimensional Harmonic oscillator:

\[
\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \quad \hat{a} = \frac{1}{\sqrt{2m\omega}} (\hat{p} - i m \omega \hat{x}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega}} (\hat{p} + i m \omega \hat{x}),
\]

\[ [\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{H}_0, \hat{a}^\dagger] = \hbar \omega \hat{a}^\dagger, \]

\[ \hat{a}|n\rangle = \sqrt{n}|n - 1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle \]

Time-independent Perturbation Theory:

\[ E_{n}^{1} = \langle n|\hat{H}_p|n\rangle; \quad E_{n}^{i} = \langle p_{n}^{0}|\hat{H}_p|p_{n}^{i-1}\rangle; \quad |p_{n}^{1}\rangle = \sum_{k \neq n} \frac{\langle k|\hat{H}_p|n\rangle}{E_{n}^{0} - E_{k}^{0}} |k\rangle \]

Angular Momentum Operators:

\[ [J_x, J_y] = i\hbar J_z \quad \text{(or)} \quad [J_i, J_j] = i\hbar \epsilon_{ijk} J_k \]

\[ J^2, J_i = 0 \quad J_{\pm} = J_x \pm iJ_y, \]

\[ J_z|j m\rangle = \hbar m |j m\rangle \quad J^2|j m\rangle = \hbar^2 j(j + 1) |j m\rangle, \]

\[ J_{\pm}|j m\rangle = \hbar \sqrt{j(j + 1) - m(m \pm 1)} |j m \pm 1\rangle \]
Qualifying Exam
Quantum Mechanics
June 2018

Solve one of the two problems in Part A, and one of the two problems in Part B.
Each problem is worth 50 points.

Part A

Problem 1. Consider two observables \( \hat{A} \) and \( \hat{B} \) in a three-dimensional Hilbert space.
In the basis:

\[
|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

the observables \( \hat{A} \) and \( \hat{B} \) are represented, respectively, by the matrices

\[
A \rightarrow \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad B \rightarrow \begin{pmatrix} 2 & b_1 & 0 \\ b_1 & 2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}
\]

where the parameters \( a_1, a_3, b_1, b_3 \) are real numbers, and \( a_1 \neq a_3 \).

a) [5 points] Show that the observables \( \hat{A} \) and \( \hat{B} \) are compatible.

b) [15 points] Find a common set of eigenstates, and list the corresponding eigenvalues.

c) [10 points] Do \( \hat{A} \) and \( \hat{B} \) form a complete set of compatible observables for all values of the parameters? Which values of the parameters should be excluded?

Now consider a quantum system in the state \( |s\rangle = \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle \).

d) [10 points] Find the probabilities of the possible results of a measurement of \( \hat{A} \) in the state \( |s\rangle \). Compute the expectation value of \( \hat{A} \) in the state \( |s\rangle \).

e) [10 points] Repeat the calculations of part d) for the observable \( \hat{B} \).

Problem 2. Consider a particle of mass \( m \) confined in a three-dimensional rectangular box with impenetrable walls. The widths \( a_i \) of the box in the three directions are such that \( a_1 = a_2 \neq a_3 \). The potential can be written as \( V(x_1, x_2, x_3) = \sum_{i=1}^{3} V_i(x_i) \) where

\[
V_i(x_i) = \begin{cases} \infty & x_i \leq 0, \ x_i \geq a_i \\
0 & 0 < x_i < a_i \end{cases}
\]

a) [5 points] Show that the problem is separable and admits factorized solutions.
b) [15 points] What are the energy levels of the system? Is there a degeneracy? *Hint:* Prove that the energy spectrum for the one-dimensional well of width $a$ is given by $E_n = \frac{\hbar^2 \pi^2}{2ma^2}$ with $n = 1, 2, \ldots$.

c) [10 points] Find a complete set of eigenfunctions of the Hamiltonian (you don’t need to normalize them).

d) [5 points] Choose $a_3 = 2a_1$. Compute the degeneracies of the first three energy levels (ground state, first and second excited states) and the energy gaps between them.

Take the limit $a_3 >> a_1$. In this limit, the box becomes an infinite rectilinear guide with a cross section that is a square of edge $a_1$.

e) [10 points] Write an expression for the eigenfunctions and the energy spectrum of the system (again, ignore normalization).

f) [5 points] What is the minimum energy (called the threshold energy) the particle must have in order to propagate along the direction $x_3$?
Qualifying Exam Quantum Mechanics

Part B

Problem 3.

A flat layer of Aluminum Gallium Arsenide (ALGaAs) of width $L$ is between two bulk pieces of Gallium Arsenide (GaAs). The potential profile of the conduction band is shown in the figure below:

The Fermi level of conduction electrons in GaAs is $E_F$. The difference in conduction band heights between ALGaAs and GaAs produces a potential barrier of $V_0$. The effective mass of electrons in GaAs is $m^*$.

(a) In the high and wide barrier limit of the WKB approximation, the transmission coefficient is given by

\[ T = \exp\{-2\gamma\}, \]

where \( \gamma = \frac{\sqrt{2m^*}}{\hbar} \int_{x_1}^{x_2} \sqrt{V(x) - E} \, dx \).

Using this approximation, determine the expression for the tunneling probability (10 points).

(b) Estimate the value of the tunneling probability for $E_F = 0.05$ eV, $V_0 = 0.3$ eV, $L = 10$ nm, $m^* = 0.067 \, m_0$, where $m_0 = 9.1 \times 10^{-31}$ kg is the free electron mass, $\hbar = 6.626 \times 10^{-34}$ m$^2$ kg / s, electron charge $e = 1.6 \times 10^{-19}$ C (10 points).

(c) Next, a voltage $V_a = 0.01$ V is applied across the barrier. Sketch the resultant change of the conduction-band edge as a function of position $x$ (5 points). Repeat the calculations you made in Part (a) and Part (b) (20 points). Discuss how the result for the tunneling probability is affected (5 points).
Problem 4.

Consider the three one-dimensional systems: (i) free particle, (ii) a particle in the potential \( V = -F_0x \), where \( F_0 \) is a constant, and (iii) harmonic oscillator of frequency \( \omega \). For each of these systems, do the following:

(a) write and solve the Heisenberg equations of motion for the coordinate and momentum operators (10 points);

(b) for the state \( \Psi(x) = \frac{1}{\sqrt{a\sqrt{\pi}}} \exp\{ip_0x/\hbar - (x - x_0)^2/2a^2\} \), find the expectation values \( \langle \hat{x}(t) \rangle, \langle \hat{p}(t) \rangle, \langle \hat{x}^2(t) \rangle \), and \( \langle \hat{p}^2(t) \rangle \) (20 points).

(c) for the state above, determine the uncertainties \( \Delta x^2 \) and \( \Delta p^2 \) (5 points);

(d) calculate the commutator of the coordinate and momentum operators at different times \( t \) and \( t' \), that is \( [\hat{x}(t),\hat{p}(t')] \) (15 points).

Useful integrals:

\[
\int_{-\infty}^{\infty} \exp\left\{-\frac{(x-x_0)^2}{a^2}\right\} \ dx = a\sqrt{\pi} ; \\
\int_{-\infty}^{\infty} x \exp\left\{-\frac{(x-x_0)^2}{a^2}\right\} \ dx = a\sqrt{\pi}x_0 ; \\
\int_{-\infty}^{\infty} x^2 \exp\left\{-\frac{(x-x_0)^2}{a^2}\right\} \ dx = a\sqrt{\pi}\left(x_0^2 + \frac{a^2}{2}\right). 
\]
Solve one of the two problems in Part A, and one of the two problems in Part B. Each problem is worth 50 points.

Part A

Problem 1. Consider the infinite one-dimensional potential well, described by \( \hat{H}_0 = \frac{\hat{p}^2}{2m} + V(x) \), where

\[
V(x) = \begin{cases} 
0 & -\frac{a}{2} \leq x \leq \frac{a}{2} \\
\infty & \text{otherwise}
\end{cases}
\]

In this problem we consider the effect of a small perturbation

\[
V_\lambda(x) = \begin{cases} 
-\lambda & x < 0 \\
\lambda & x > 0 
\end{cases}
\]

a) [15 points] Show that the normalized eigenfunctions of the unperturbed Hamiltonian \( \hat{H}_0 \) are

\[
\psi_n^0(x) = \begin{cases} 
\sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right) & n = 2, 4, \ldots \\
\sqrt{\frac{2}{a}} \cos \left( \frac{n\pi x}{a} \right) & n = 1, 3, \ldots
\end{cases}
\]

with corresponding eigenvalues \( E_n^0 = \frac{\pi^2 \hbar^2 n^2}{2ma^2} \).

b) [10 points] Find the corrections \( E_n^1 \) to the energy eigenvalues, due to the term \( V_\lambda(x) \), to first order in perturbation theory.

c) [25 points] Find the first order corrections \( \psi_n^1(x) \) to the energy eigenstates. Use these to compute the second order corrections \( E_n^2 \) to the energy levels.

Problem 2. Consider a particle of spin \( s = 1 \), placed in a uniform magnetic field. This is described by the Hamiltonian \( \hat{H} = -\alpha \vec{B} \cdot \hat{\vec{S}} \). At time \( t = 0 \), the system is in the physical state represented by

\[
|\chi_0\rangle = \sqrt{\frac{1}{3}} |1\rangle + \sqrt{\frac{2}{3}} |0\rangle,
\]

where \( |1\rangle \), \( |0\rangle \), and \( |-1\rangle \) are the three eigenstates of \( \hat{S}_z \).

a) [15 points] From the properties of the angular momentum operators (shown in the formulas below) construct explicitly the matrix representations of the operators \( \hat{S}_z, \hat{S}_x, \hat{S}_y, \hat{S}_z \) in the basis of eigenstates of \( \hat{S}_z \), for the case \( s = 1 \).
b) [5 points] Choose the magnetic field along the z-axis, and write the matrix representation for \( \hat{H} \). List the eigenvalues and eigenstates of \( \hat{H} \).

c) [5 points] What are the possible outcomes of a measurement of the energy in the state \( |\chi_0\rangle \) at time \( t = 0 \)? What is the probability of each of these outcomes?

d) [5 points] Compute the expectation values of \( \hat{H} \) and \( \hat{S}_x \) at time \( t = 0 \).

e) [20 points] Consider the state \( |\chi(t)\rangle \) of the system at time \( t \). List the possible outcomes of a measurement of the energy at time \( t \). What is the expectation value of the energy at time \( t \)? What is the expectation value of \( \hat{S}_x \) at time \( t \)?

Formulas for Part A

Schrödinger Equation:

\[
i\hbar \frac{\partial \psi(x,t)}{\partial t} = \hat{H}\psi(x,t) , \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(x) , \quad \hat{H}\psi_E(x) = E\psi_E(x)
\]

Time-independent Perturbation Theory:

\[
E^1_n = \langle n|\hat{H}_p|n\rangle; \quad E^i_n = \langle p^0_n|\hat{H}_p|p_n^{i-1}\rangle; \quad |p^1_n\rangle = \sum_{k \neq n} \frac{\langle k|\hat{H}_p|n\rangle}{E^1_n - E^i_k} |k\rangle
\]

Angular Momentum Operators:

\[
[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \quad \text{(or } [\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k), \quad [\hat{J}^2, \hat{J}_i] = 0, \quad \hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \quad \hat{J}_\pm |j \ m\rangle = \hbar \sqrt{j(j+1)-m(m\pm1)} |j \ m\pm1\rangle
\]
Qualifying Exam
Quantum Mechanics, Part B

Problem 3.
The Hamiltonian of an electron in the presence of a magnetic field in arbitrary direction is given by
\( \hat{H} = \left( \hat{p} - e\hat{A}/c \right)^2 / 2m \), where \( \hat{A} = \hat{A}(\hat{r}) \) is a position-dependent vector potential.

a. Using the Heisenberg equations of motion, obtain the expressions for the components of velocity operator, \( \hat{v} = d\hat{r}/dt \). (15 points)

b. Calculate the commutators of these components, \([\hat{v}_x, \hat{v}_x], [\hat{v}_x, \hat{v}_z], \) and \([\hat{v}_y, \hat{v}_z] \). (20 points)

c. Calculate the commutators of these components with the components of the coordinate operator, \([\hat{v}_x, \hat{x}], [\hat{v}_x, \hat{y}], [\hat{v}_x, \hat{z}], [\hat{v}_y, \hat{x}], [\hat{v}_y, \hat{y}], [\hat{v}_y, \hat{z}], [\hat{v}_z, \hat{x}], [\hat{v}_z, \hat{y}], \) and \([\hat{v}_z, \hat{z}] \). (15 points)

Useful formulas:
\([\hat{x}, f(\hat{p}_x)] = i\hbar \partial f / \partial \hat{p}_x \) and \([\hat{p}_x, g(\hat{x})] = -i\hbar \partial g / \partial \hat{x} \), where \( f \) and \( g \) are arbitrary functions.
Problem 4.
Consider a one-dimensional harmonic oscillator of mass \( M \) and frequency \( \omega \). The mass has a charge \( q \) and is in an external time-dependent electric field \( \varepsilon(t) \). The electric field gives a contribution to the potential \( \hat{H}^{(t)} = q \hat{x} \varepsilon(t) \).

a. Writing the position operator in terms of the creation and annihilation operators as
\[
\hat{x} = \sqrt{\frac{\hbar}{2M\omega}}(\hat{a}^+ + \hat{a}),
\]
show that the transition matrix elements are
\[
H_{mn}^{(t)}(t) = q \varepsilon(t) \sqrt{\frac{\hbar}{2M\omega}} \left( \sqrt{n} \delta_{m,n-1} + \sqrt{n+1} \delta_{m,n+1} \right).
\] (10 points)

b. In time-dependent perturbation theory, the transition probability to first-order is given by
\[
W^{(1)}(n \rightarrow m) = \frac{1}{\hbar} \left| \int_{-\infty}^{\infty} dt \, H_{mn}^{(t)}(t) \exp\left\{ -\frac{i}{\hbar} \left( E_m - E_n \right) t \right\} \right|^2.
\]
Determine all non-vanishing transition probabilities from the \( n \)-th state, for the cases
i) \( \varepsilon(t) = \varepsilon_0 \tau \delta(t - t_0) \); (10 points)
ii) \( \varepsilon(t) = \varepsilon_0 \exp\left\{ -t^2 / \tau^2 \right\} \); (10 points)
iii) \( \varepsilon(t) = \varepsilon_0 \frac{1}{1 + t^2 / \tau^2} \). (10 points)

c. Take a ratio to compare the probabilities of the transitions for the cases ii) and iii) with \( \tau = 10 \) ns and \( \omega/2\pi = 10 \) GHz (5 points). Make the same comparison for \( \omega/2\pi = 10 \) kHz (5 points).

Useful integrals:
\[
\int_{-\infty}^{\infty} dt \exp\left\{ i \omega t - \frac{t^2}{\tau^2} \right\} = \sqrt{\pi} \tau \exp\left\{ -\omega^2 \tau^2 / 4 \right\}
\]
\[
\int_{-\infty}^{\infty} dt \exp\left\{ i \omega t \right\} \frac{1}{1 + t^2 / \tau^2} = \pi \tau \exp\left\{ -|\omega\tau| \right\}
\]
Qualifying Exam: Quantum Mechanics (June 2019)
Ph.D. Program in Physics, The Graduate Center, CUNY

○ Solve 2 out of the following 3 problems.
○ Indicate clearly which two problems you choose to solve, and show all calculations.

Problem 1.
An electron of mass \( \mu \) and charge \( e \) is confined to a ring of radius \( a \), so its momentum operator is given by
\[
\hat{p} = -\frac{i\hbar}{a} \frac{d}{d\varphi}.
\]

a. What is the Hamiltonian of this system (5 points)?

b. Write down and solve the time-independent Schrödinger equation for the particle (10 points).

c. Using periodic boundary conditions and normalization, show that the eigenvectors are given by
\[
|m\rangle = \frac{1}{\sqrt{2\pi}} e^{im\varphi},
\]
where \( m = 0, \pm 1, \pm 2, \ldots \) and determine the eigenenergies of the system (15 points). What are the degeneracies of the energy levels (5 points)?

d. Using time-independent perturbation theory, calculate the first-order corrections to the energy eigenvalues,
\[
E_m^{(1)} = \langle m|\hat{V}|m\rangle (5 \text{ points}),
\]
due to an electric field applied along the \( x \)-direction, for which the perturbation potential is
\[
\hat{V} = -e\mathcal{E}x = -e\mathcal{E}a \cos \varphi.
\]
Calculate the second-order corrections,
\[
E_m^{(2)} = \sum_{k \neq m} |\langle k|\hat{V}|m\rangle|^2 \left/ \left( E_m^{(0)} - E_k^{(0)} \right) \right. (10 \text{ points}),
\]
due to the same potential.

Problem 2.
A spin one-half particle is in a one-dimensional harmonic potential of frequency \( \omega \).
Initially, it is in a state
\[
|\alpha\rangle = \frac{1}{\sqrt{2}} (|0\rangle \uparrow + |1\rangle \downarrow),
\]
where spin-up and spin-down states are defined with respect to the \( z \)-projection of the spin operator and ket-vectors, \( |0\rangle \) and \( |1\rangle \), correspond the ground state and the first excited state of the harmonic oscillator, respectively. At time \( t = 0 \), a uniform magnetic field along the \( z \)-axis is turned on, providing the additional energies of \( +\mu_B B \) and \( -\mu_B B \) for the spin-up and spin-down states, respectively. Here, \( \mu_B \) is the Bohr magneton.

a. Calculate the energy expectation value for the system with magnetic field (10 points).

b. Determine the time evolution of \(|\alpha\rangle\) (15 points).

c. Calculate the expectation values for the coordinate and momentum operators at time \( t > 0 \). (25 points).
Problem 3.
An electron of mass $\mu$ and charge $e$ in a one-dimensional potential well of size $L$ and infinite walls is excited to the third energy level and allowed to return to the ground state by way of spontaneous emission. This can happen in one of two ways: $|3\rangle \rightarrow |1\rangle$, and $|3\rangle \rightarrow |2\rangle \rightarrow |1\rangle$, where the notation $|n\rangle$ is used to designate the state belonging to the corresponding energy level.

a. At which frequencies can the spontaneously emitted light be observed (10 points)?

b. What is the spontaneous emission rate for each frequency (30 points)?

Recall that $W_{i\rightarrow f}^{em} = \frac{4\omega_i^3}{3\hbar c^3} |d_{fi}|^2$, where

$$d_{fi} = e \int \psi^*_f(x) \hat{x} \psi_i(x) dx$$

is the matrix element of the electric dipole.

c. Find the rate at which the number of atoms in the ground state increases (10 points).

Hint: remember that if a random event is a sequence of two independent events occurring subsequently (one after the other), the total probability is the product of the probabilities of the individual events. If, however, an event can occur via different processes, the total probability is the sum of probabilities of each alternative.

Useful formulas:

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b)$$

$$\int_c^d x \cos(ax) dx = \frac{x \sin(ax)}{a} \bigg|_c^d + \frac{\cos(ax)}{a^2} \bigg|_c^d$$
Qualifying Exam
Quantum Mechanics
January 2020

Solve one of the two problems in Part A, and one of the two problems in Part B.
Each problem is worth 50 points.

Part A

Problem 1.
Consider a system of two interacting spins described by Hamiltonian

$$\hat{H} = a S_{1z} + b S_{2z} + c S_1 \cdot S_2,$$

where $S_i = (S_{ix}, S_{iy}, S_{iz})$. Define $J = S_1 + S_2$ the total angular momentum of the system.

a) [10 points] The system can be described either using the basis $|m_1, m_2\rangle \equiv |s_1 s_2 m_1 m_2\rangle$ (eigenstates of $S_1^2, S_2^2, S_1z,$ and $S_2z$) or the basis $|j m_j\rangle \equiv |s_1 s_2 j m_j\rangle$ (eigenstates of $S_1^2, S_2^2, J^2,$ $J_z$). Show that, for a generic choice of the parameters $a$, $b$, and $c$, the Hamiltonian $\hat{H}$ is non-diagonal, in both of these bases.

b) [20 points] Let us set $a = b$. Find the exact energy eigenvalues of the system in the general case (without choosing specific values for $s_1$ and $s_2$).

Consider now the case $s_1 = s_2 = 1/2$.

c) [10 points] Find the expression of the states $\{|j m_j\rangle\}$ in terms of the appropriate elements of the basis $\{|m_1 m_2\rangle\}$. This part requires the direct construction of all CG coefficients, namely to prove that

$$|1, 1\rangle = |+, +\rangle, \quad |1, 0\rangle = \frac{1}{\sqrt{2}}(|+, -\rangle + |-, +\rangle), \quad |0, 0\rangle = \frac{1}{\sqrt{2}}(|+, -\rangle - |-, +\rangle), \quad |1, -1\rangle = |-, -\rangle$$

(where we adopted the usual simplified notation $|+, +\rangle \equiv |1/2, +1/2\rangle$, etc...).

d) [10 points] Write the corresponding energy spectrum, as derived in part b), and comment on its degeneracy.

Problem 2.
Consider a three-state quantum system described by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_p$. where

$$\hat{H}_0 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad \hat{H}_p = x \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We want to apply perturbation theory to study the energy levels of the system. Let us begin with the non-degenerate case $a_1 = 1$ eV, $a_2 = 2$ eV, $a_3 = 3$ eV (i.e., $a_1 \neq a_2 \neq a_3$):
a) [5 points] What are the conditions that the parameter $x$ should satisfy in order for perturbation theory to be applicable?

b) [10 points] After writing down the eigenstates and eigenvalues of $\hat{H}_0$, compute the first- and second-order corrections to the energy levels due to $\hat{H}_p$.

c) [10 points] Compute the energy eigenstates of $\hat{H}$ up to first order in perturbation theory.

d) [10 points] Compute the exact eigenvalues of $\hat{H}$. Check that, by expanding the exact result about the appropriate limit, you can reproduce the results from part b).

Let us now study the degenerate case $a_1 = 1$ eV, $a_2 = 1$ eV, $a_3 = 3$ eV (i.e., $a_1 = a_2 \neq a_3$):

e) [10 points] Compute the first-order corrections to the energy levels due to $\hat{H}_p$.

f) [5 points] Compute the energy eigenstates of $\hat{H}$ up to first order in perturbation theory.

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Formulas for Part A

Time-independent Perturbation Theory:

\[ E_n^1 = \langle n | \hat{H}_p | n \rangle; \quad E_n^i = \langle p_n^0 | \hat{H}_p | p_n^{-1} \rangle; \quad | p_n^1 \rangle = \sum_{k \neq n} \frac{\langle k | \hat{H}_p | n \rangle}{E_k - E_n} | k \rangle \]

Angular Momentum Operators:

\[ \begin{align*}
[\hat{J}_x, \hat{J}_y] &= i\hbar \hat{J}_z \quad \text{(or } [\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k), \quad [\hat{J}_z, \hat{J}_i] = 0, \quad \hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \\
\hat{J}_\pm |j \, m \rangle &= \hbar \sqrt{j(j+1) - m(m \pm 1)} |j \, m \pm 1 \rangle
\end{align*} \]
Problem 3.

An electron with charge $e = 1.6 \cdot 10^{-19}$ C and mass $m = 9.1 \cdot 10^{-31}$ kg is confined by the potential

$$V(x) = \begin{cases} 
V_0 & \text{if } x < -a \text{ or } x > a \\
-V_0 \frac{x}{a} & \text{if } -a \leq x \leq 0 \\
V_0 \frac{x}{a} & \text{if } 0 \leq x \leq a
\end{cases}$$

a. Sketch the potential and determine the turning points for an electron with energy $E < V_0$ (10 points).

b. Write the WKB quantization conditions,

$$\int_{x_1}^{x_2} dx \sqrt{2m(E - V(x))} = \pi \hbar \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2 \ldots,$$

where $x_1$ and $x_2$ are the turning points, for this specific potential (5 points).

c. Using these quantization conditions, determine the energy spectrum for this potential (20 points).

d. Estimate the number of quantized levels for $a = 8$ nm and $V_0 = 5$ eV ($\hbar = 1.05 \cdot 10^{-34}$ kg m$^2$/s) (15 points).
Problem 4.

Consider two non-interacting electrons in a one-dimensional harmonic oscillator potential along the x-axis with angular frequency $\omega$.

(a) Write down the ground state wavefunctions as products of the position and spin parts for the spin singlet (5 points) and spin triplet (5 points).

(b) For both these cases calculate the expectation values for the separation of the electrons, $\langle \hat{x}_1 - \hat{x}_2 \rangle$ (10 points), difference of their momenta, $\langle \hat{p}_1 - \hat{p}_2 \rangle$ (10 points), and variance of the separation, $\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle - \langle \hat{x}_1 - \hat{x}_2 \rangle^2$ (20 points).

Single-particle wavefunctions for the harmonic oscillator are

$$\psi_n(x) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp\left\{ -\frac{m\omega}{2\hbar} x^2 \right\} H_n \left( \sqrt{\frac{m\omega}{\hbar}} x \right), \text{ with } H_n(\xi) = (-1)^n e^{\xi^2} \left( \frac{d}{d\xi} \right)^n e^{-\xi^2}.$$ 

Useful integrals:

$$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi} ; \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} d\xi = \frac{1}{2} \sqrt{\pi} ; \int_{-\infty}^{\infty} \xi^4 e^{-\xi^2} d\xi = \frac{3}{4} \sqrt{\pi}$$